Entanglement and its key role in quantum information

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CMSA Interdisciplinary Science Seminar
June 30, 2022

HARVARD UNIVERSITY

Outline of the Presentation

- **Introduction**
  - Classical properties and states
  - Quantum properties and states
- **Entanglement**
  - Entanglement definition
  - Entanglement detection
  - Entanglement quantification
- **Applications**
  - Teleportation

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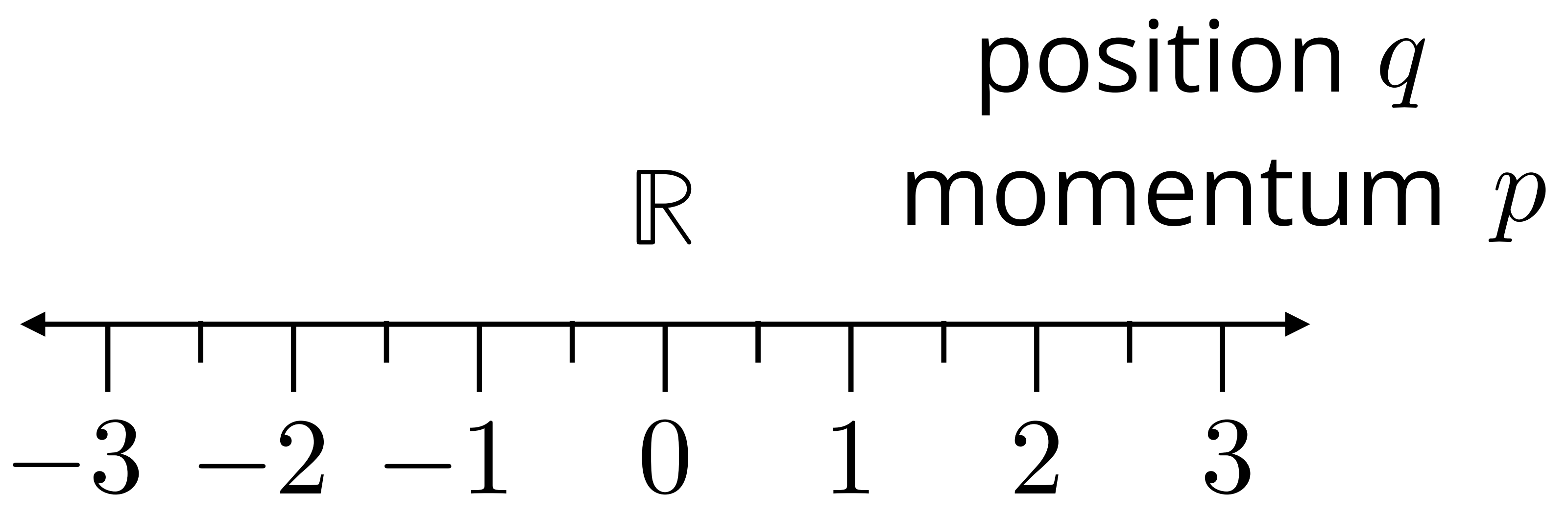
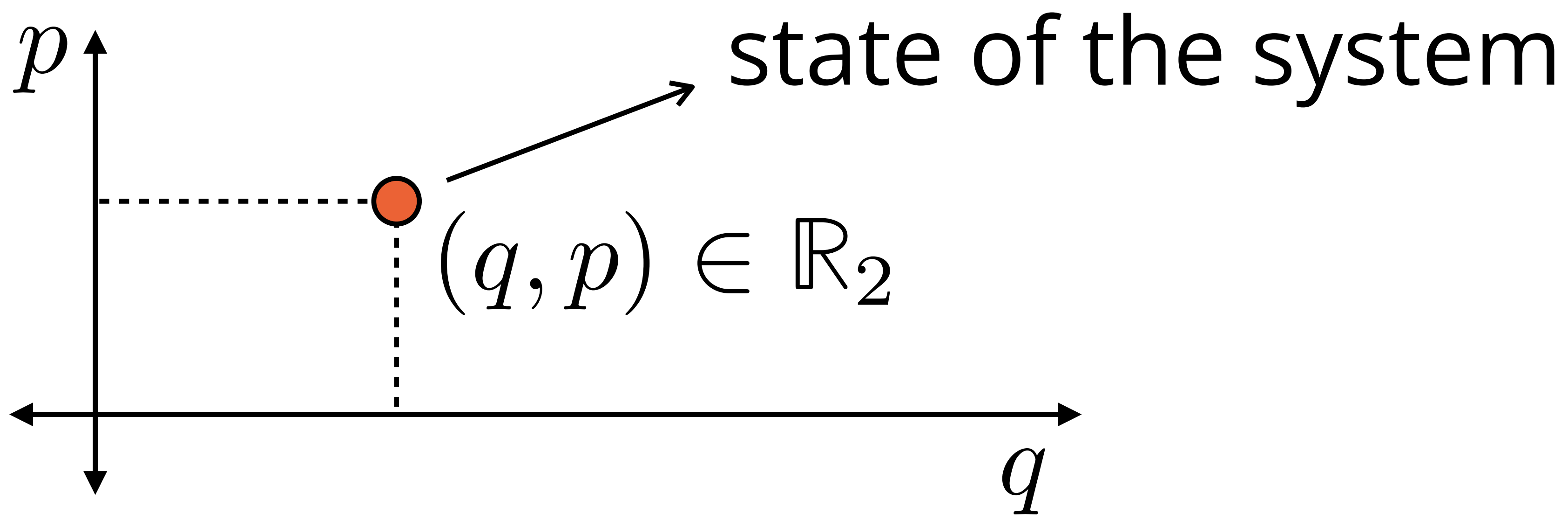
Classical Systems

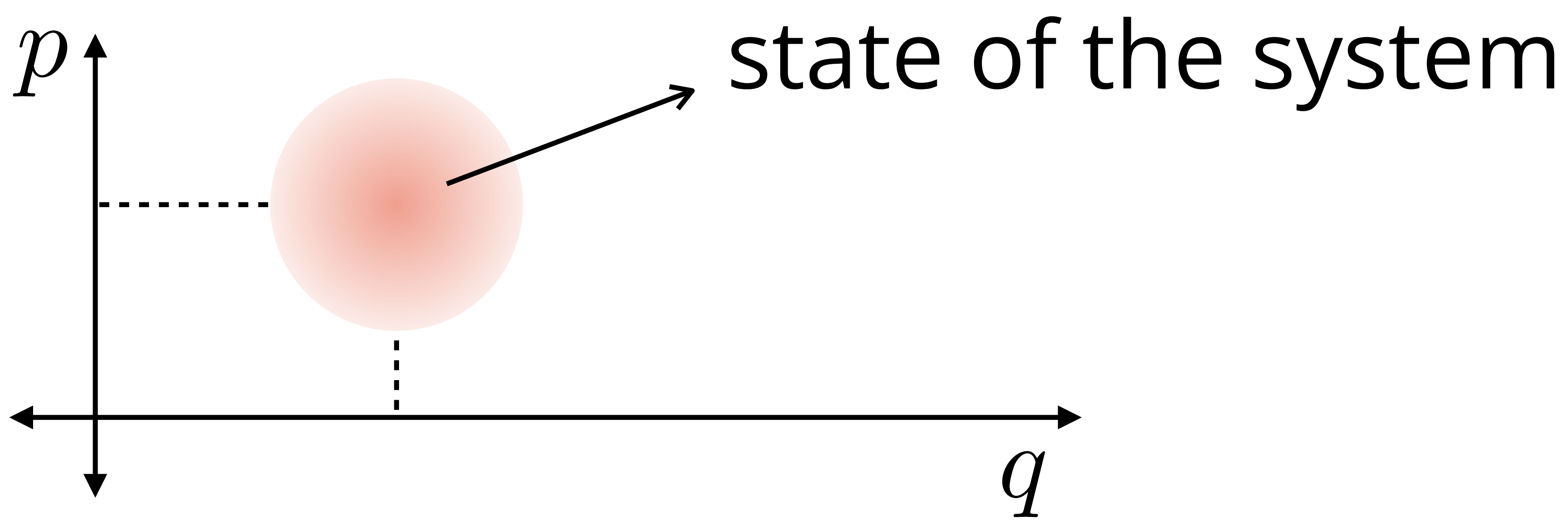
A **classical property** is represented by a **real-valued number**



A **classical system** is represented by a **point in the phase space**



A **noisy classical system** is represented by a **probability distribution** in the phase space



Dirac Notation

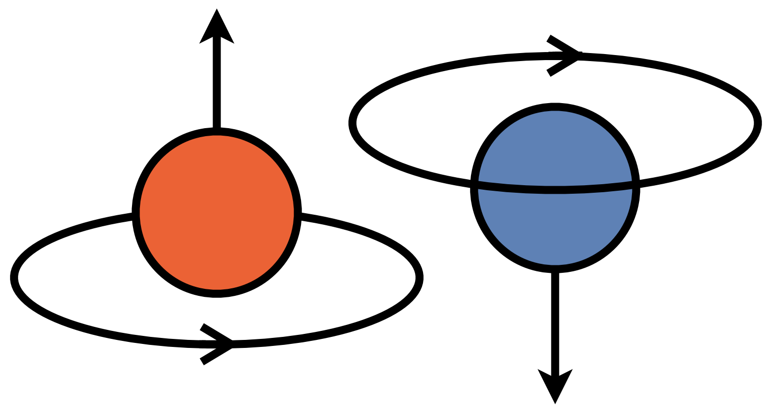
- A **column vector** is represented with a "**ket**", e.g., $|x\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$
- A **row vector** is represented with a "**bra**", e.g., $\langle y| = \begin{bmatrix} c & d \end{bmatrix}$
- A **ket** can be **transformed** into a **bra** as follows: $|x\rangle \rightarrow \langle x| = \begin{bmatrix} a^* & b^* \end{bmatrix} = |x\rangle^\dagger$ (conjugate transpose)
- The **inner product** is represented as a bra-ket $\langle y|x\rangle = \begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a \cdot c + b \cdot d$
- The **outer product** as $|x\rangle\langle y| = \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} c & d \end{bmatrix} = \begin{bmatrix} ac & ad \\ bc & bd \end{bmatrix}$ E.g., $|x\rangle\langle y| \cdot |z\rangle = |x\rangle \langle y|z\rangle = \langle y|z\rangle |x\rangle$
- We consider the **computational basis** $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Quantum Systems — Observables

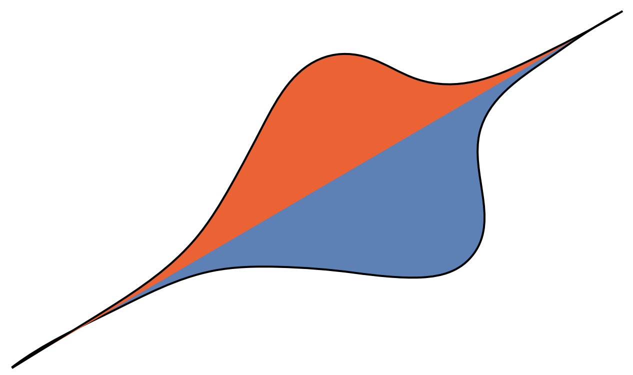
A **quantum property**, known as quantum observable, is represented by a **Hermitian matrix**

- Those matrices have **real eigenvalues** and represent the **possible outcomes of measurements**.

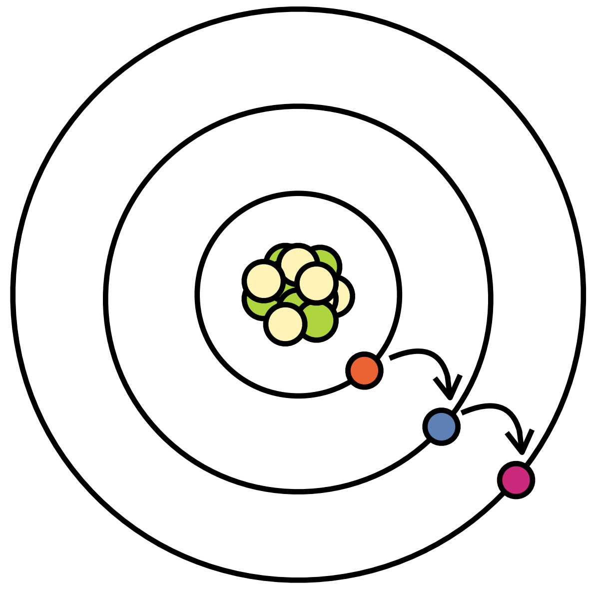
Observable with **finite** possible values $\begin{bmatrix} \star & \cdots & \star \\ \vdots & \ddots & \vdots \\ \star & \cdots & \star \end{bmatrix}$



electron spin

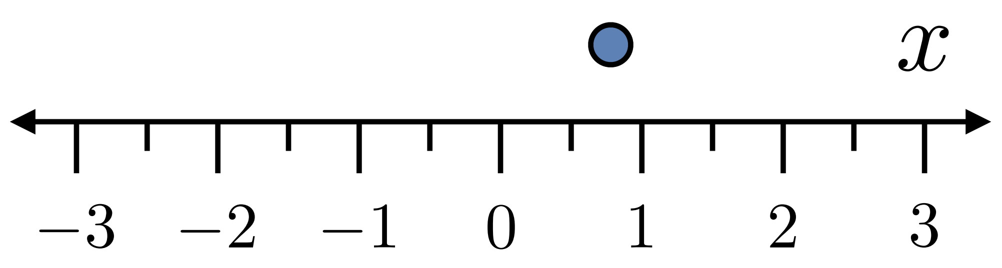


polarization

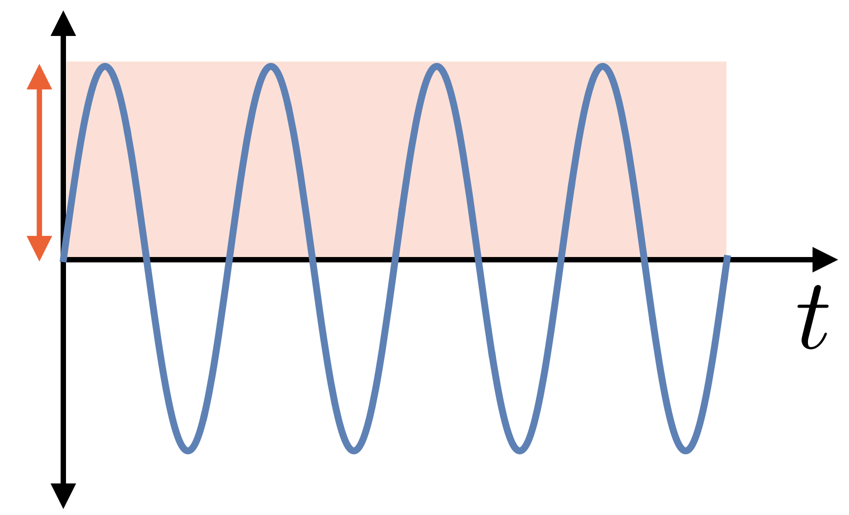


electron excitation

Observable with **infinite** possible values $\begin{bmatrix} \star & \cdots & \star & \cdots \\ \vdots & \ddots & \vdots & \cdots \\ \star & \cdots & \star & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$



position of a particle

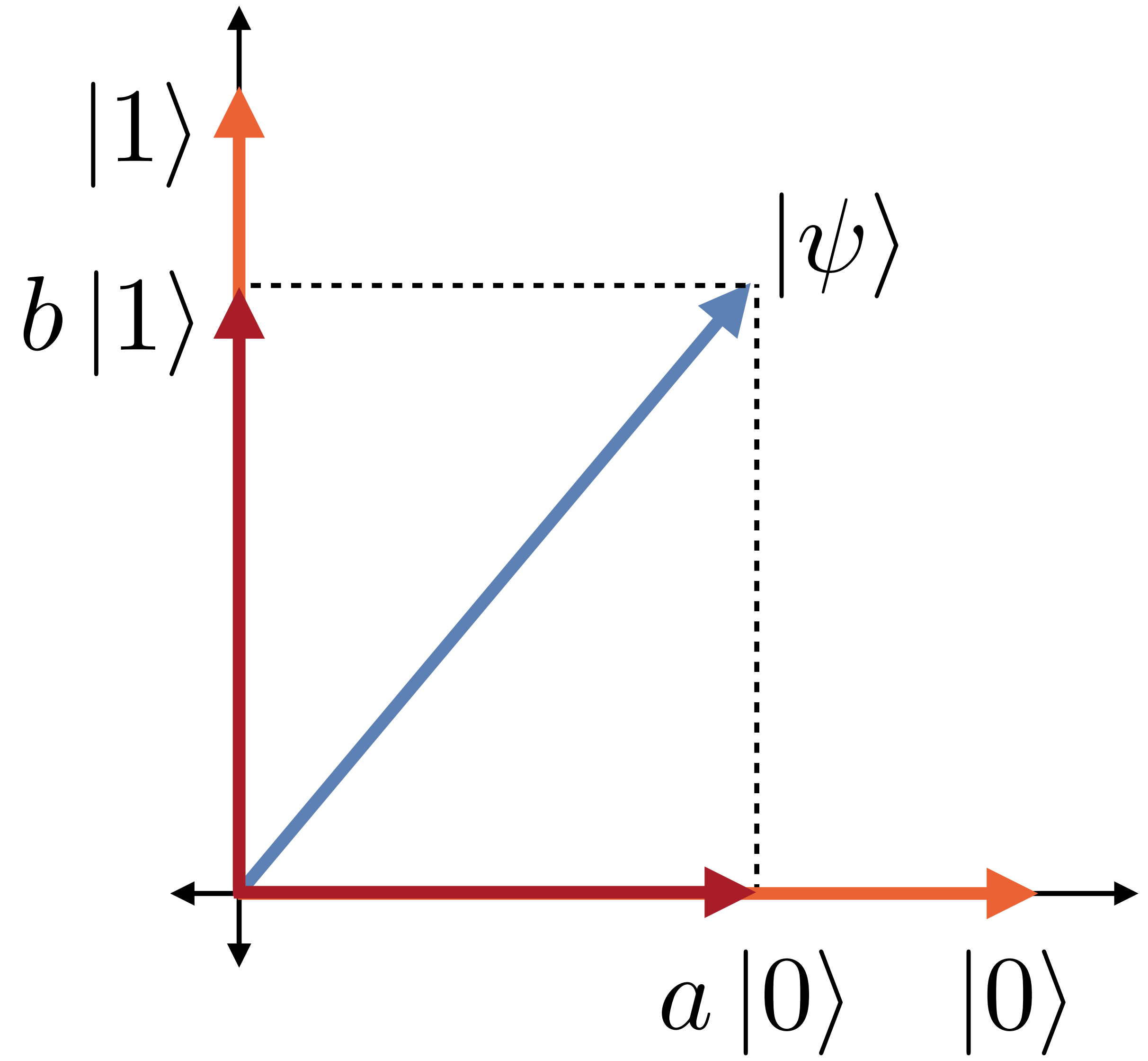


wave amplitude

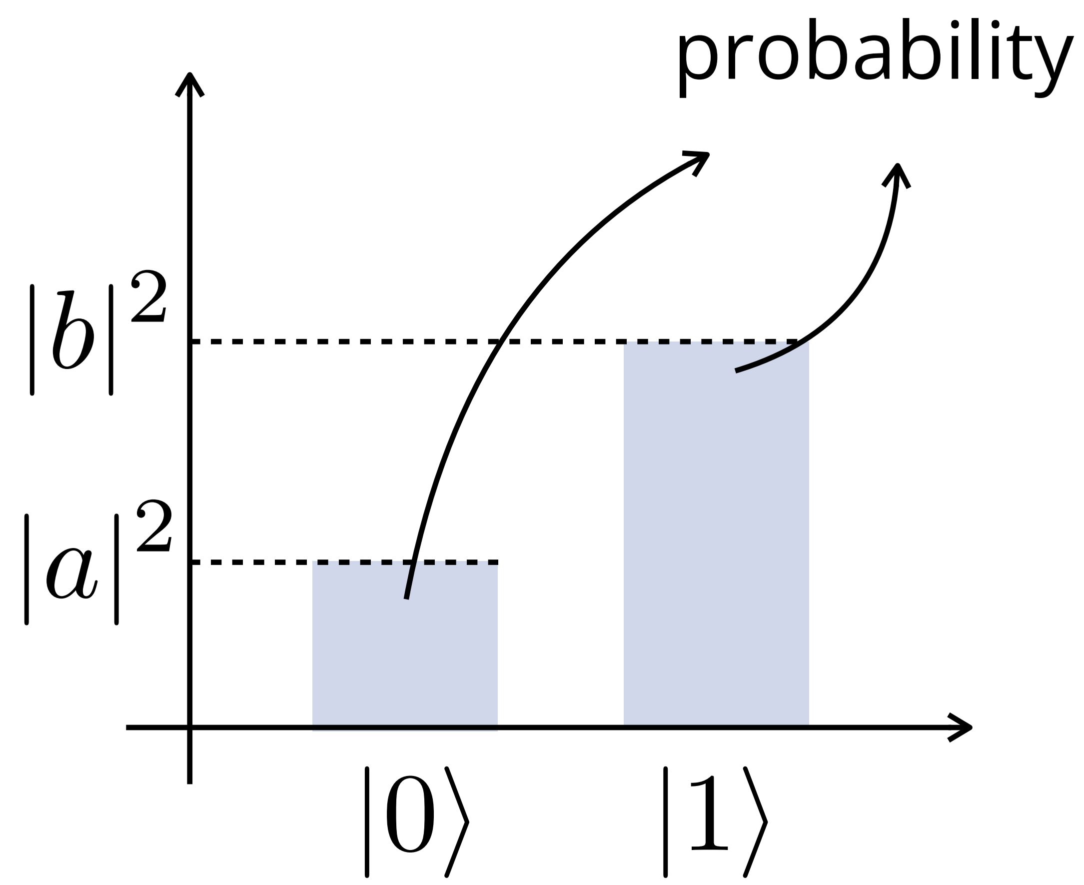
Quantum Systems — States

A **quantum system** is represented by a **normalized vector** known as the **quantum state**

State **superposition**: $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$



$$|0\rangle \perp |1\rangle \Leftrightarrow \langle 0|1\rangle$$

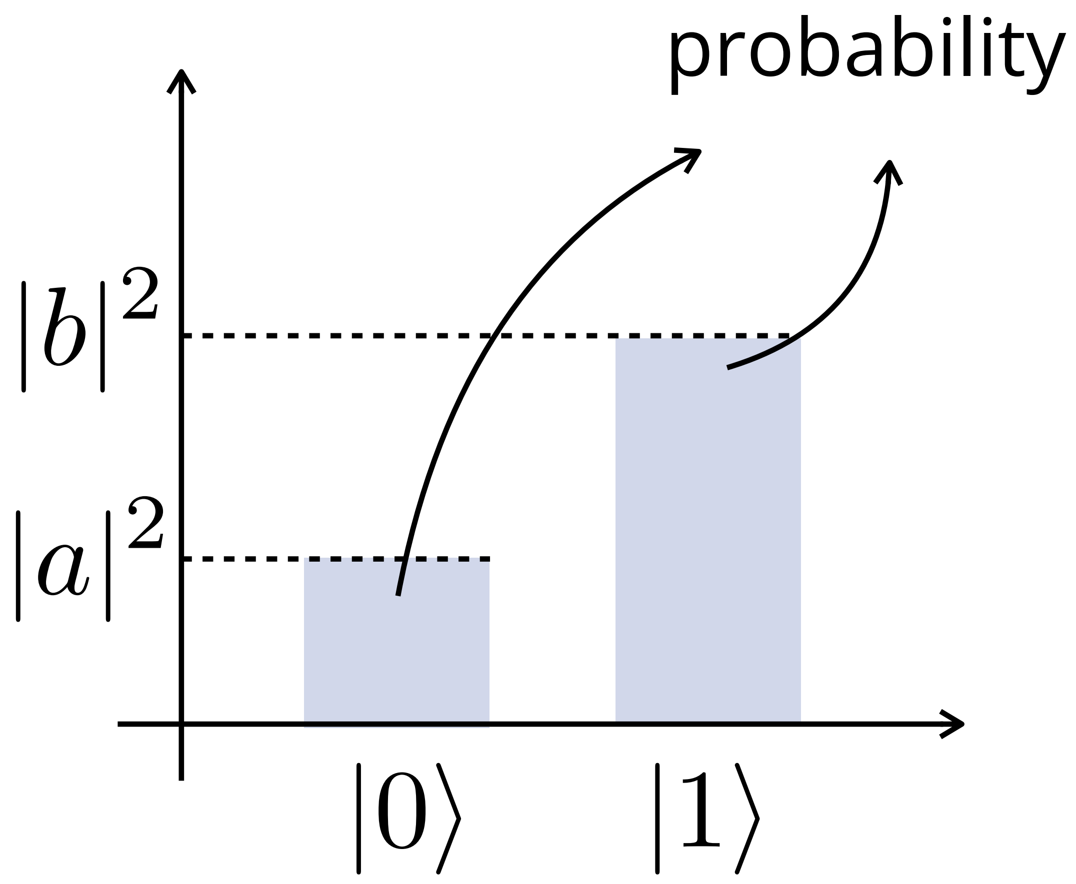


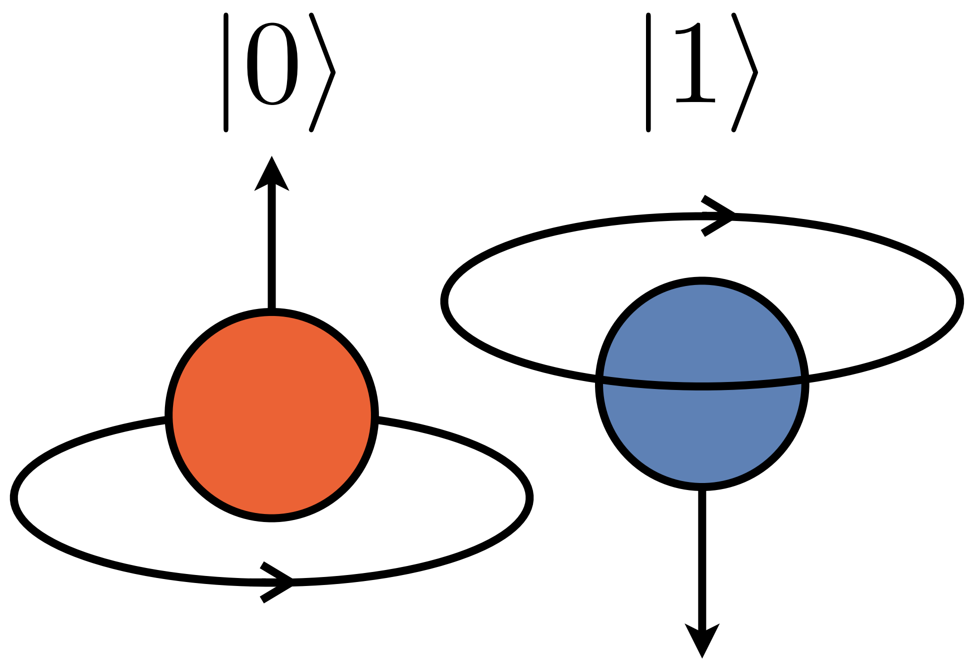
All states have the same length, i.e., normalized

$$|a|^2 + |b|^2 = 1$$

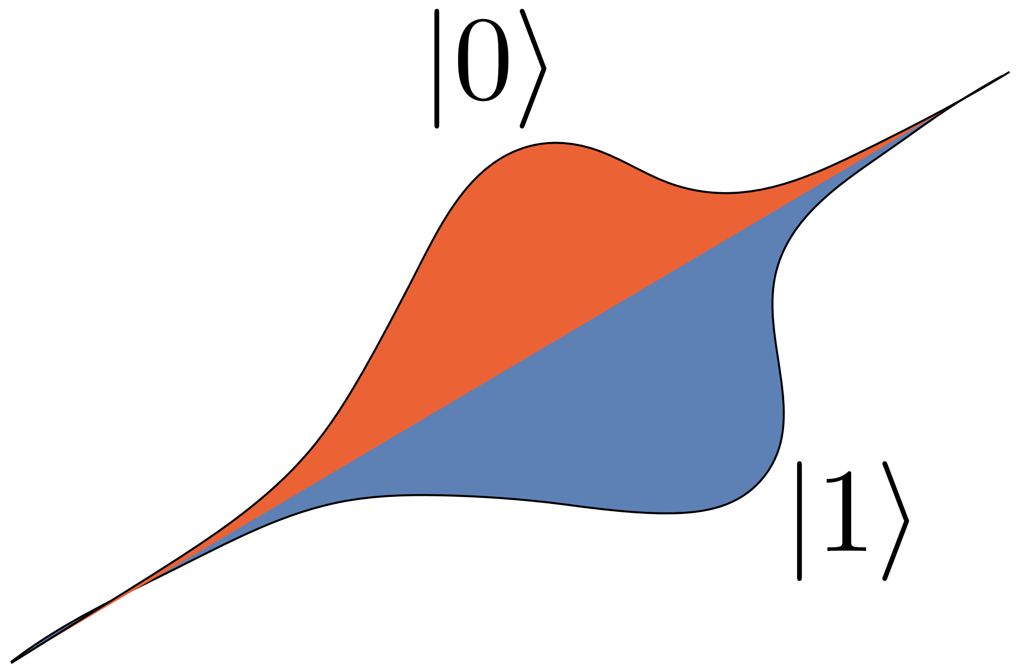
Examples of 2-dimensional States

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle$$





Electron Spin

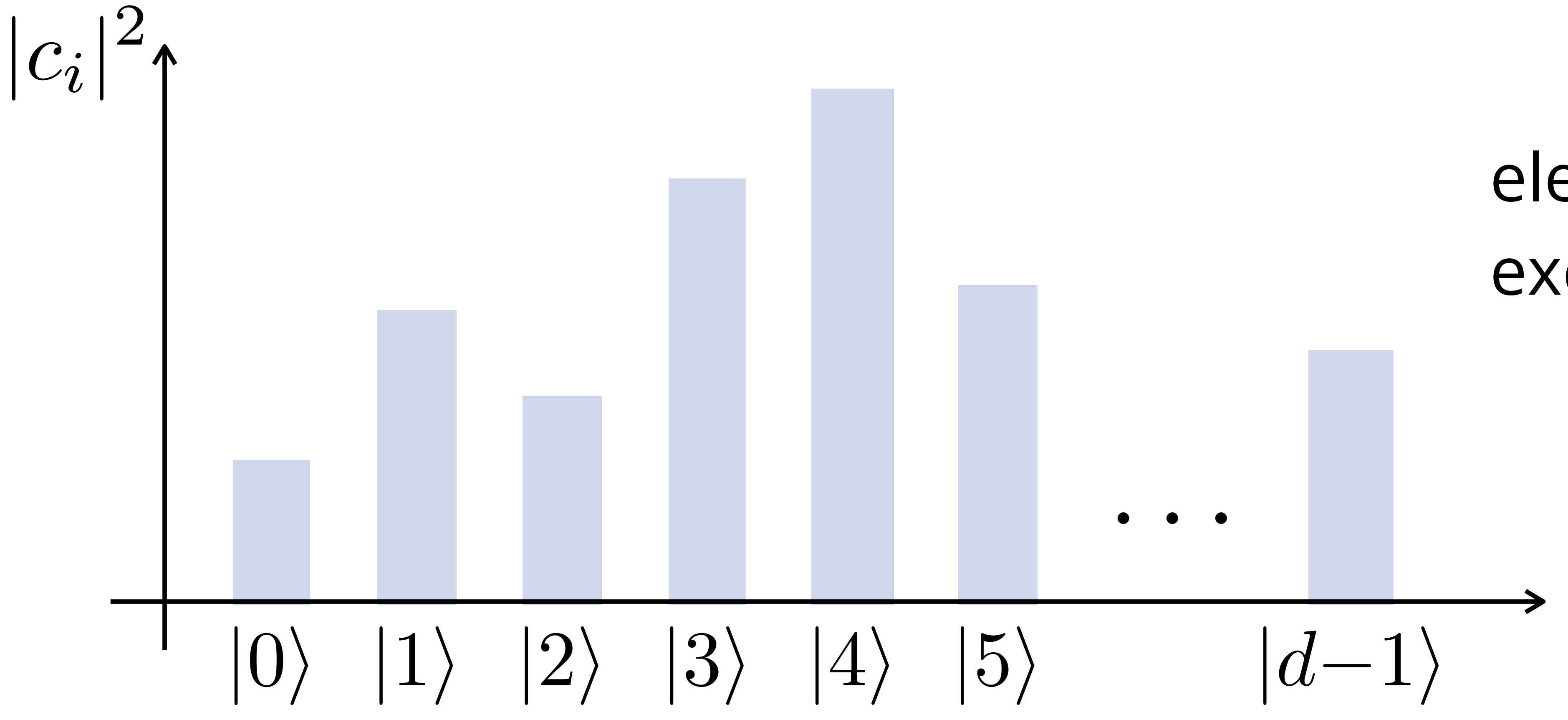


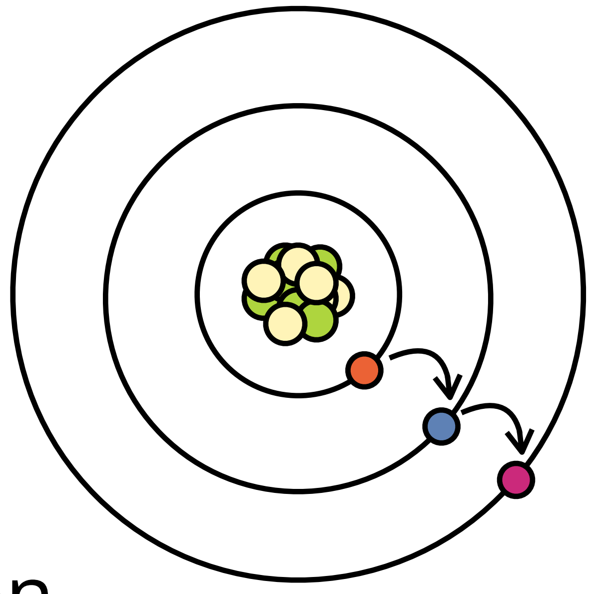
Photon Polarization

Example of n-dimensional States

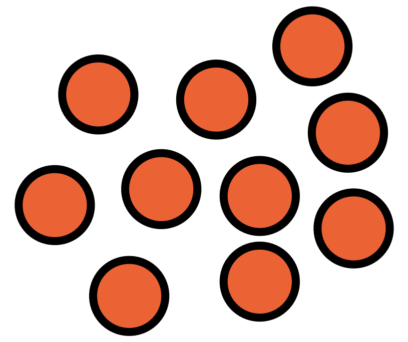
A d-dimensional quantum state

$$|\psi\rangle = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{d-1} \end{bmatrix} = \sum_{i=0}^{d-1} c_i |i\rangle$$





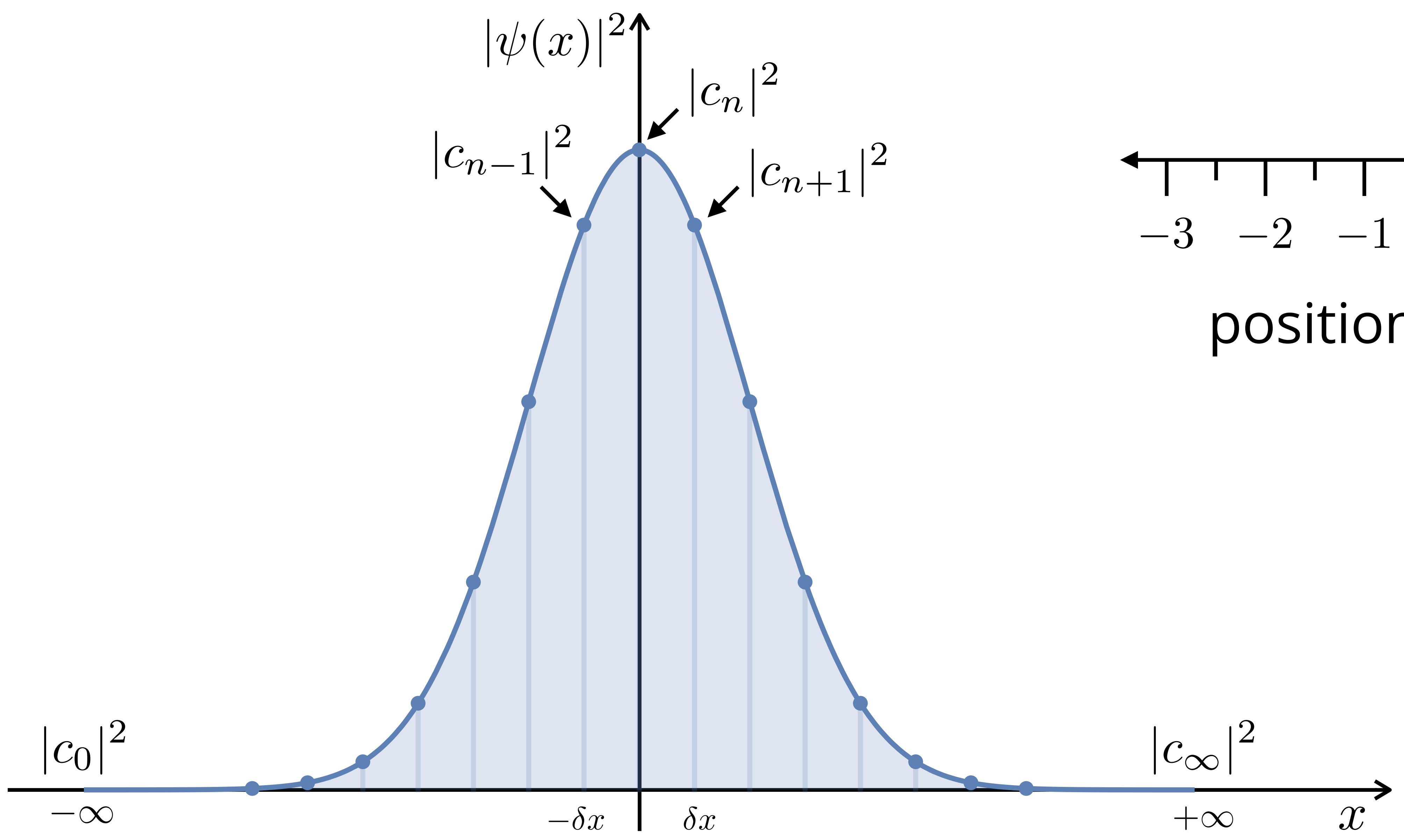
electron excitation

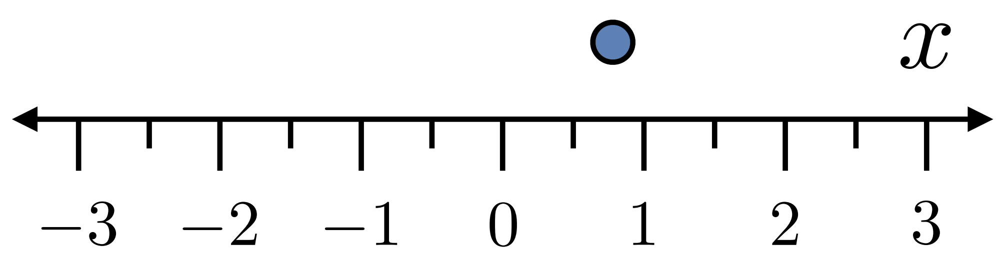


number of particles, e.g., photons

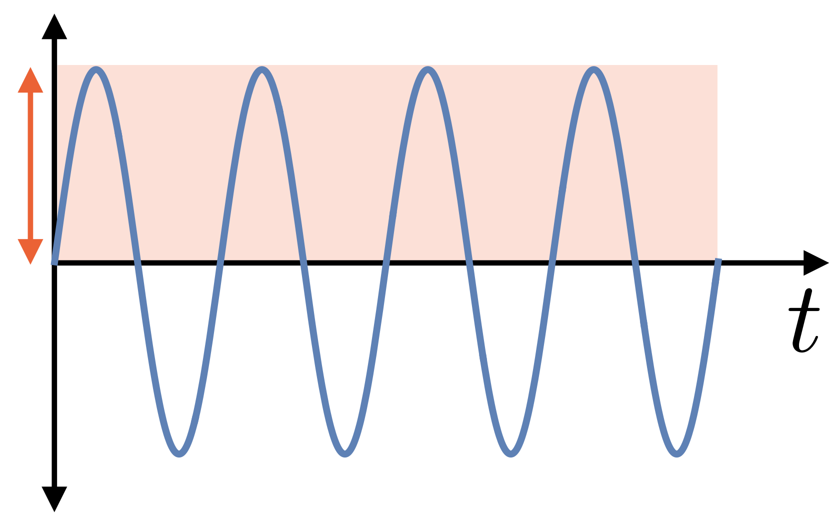
An infinite-dimensional state

$$|\psi\rangle = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \end{bmatrix} = \sum_{i=0}^{\infty} c_i |i\rangle$$





position of a particle



wave amplitude

Quantum Evolution

The **evolution** of a quantum state is described by a **unitary** transformation on the quantum state

$$|\psi\rangle \to U|\psi\rangle$$

- Unitary is a matrix that satisfies: $UU^\dagger = U^\dagger U = \mathbb{1}$ (preserves normalization)

$$X = |0\rangle\langle 1| + |1\rangle\langle 0| \qquad Y = -\imath(|0\rangle\langle 1| - |1\rangle\langle 0|) \qquad Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

- When $U = e^{\imath Ht/\hbar}$ where H is the Hamiltonian of the quantum system and $\hbar$ the reduced Planck constant the evolution is given by the **Schrödinger equation**

$$\imath\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle = H|\psi\rangle$$

Quantum Systems — States

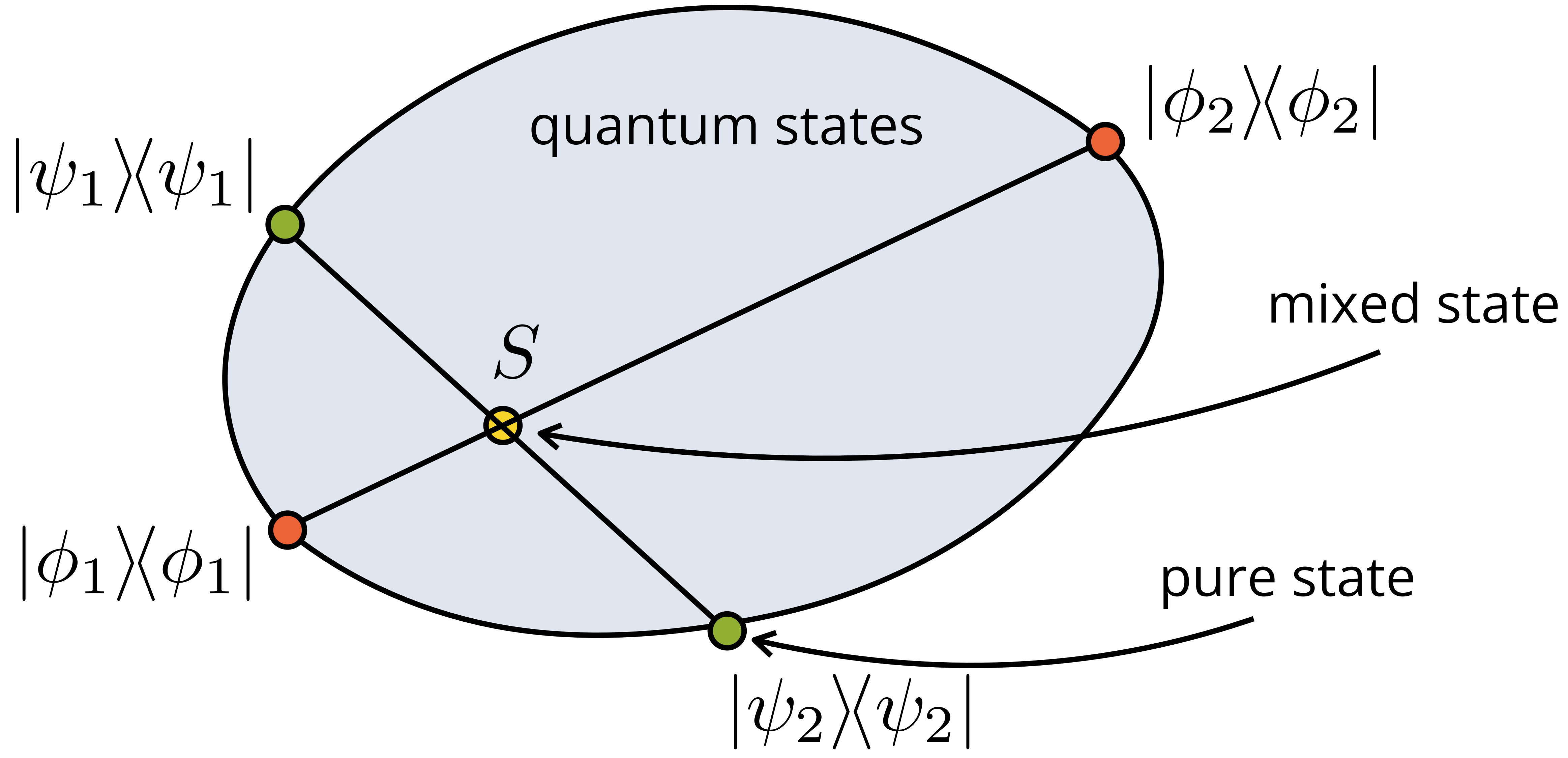
A **noisy quantum system** is represented by a positive unit-trace matrix known as the **quantum state**

- A quantum state that is an outer product of a vector is called **pure quantum state** otherwise it is called **mixed quantum state**

- The **spectral decomposition** of an arbitrary quantum state is

$$S = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|$$

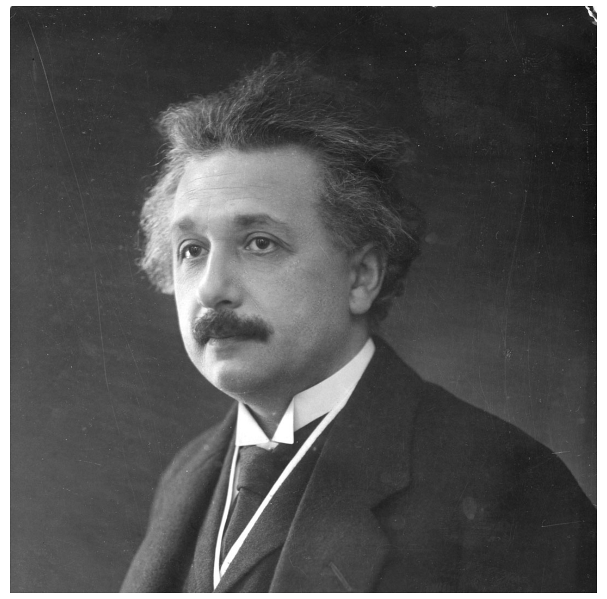
$$S = p_1 \, |\psi_1\rangle\langle\psi_1| + p_2 \, |\psi_2\rangle\langle\psi_2|$$
$$= p_1' \, |\phi_1\rangle\langle\phi_1| + p_2' \, |\phi_2\rangle\langle\phi_2|$$



- The **evolution** of a mixed quantum state is given by $S \rightarrow USU^\dagger$

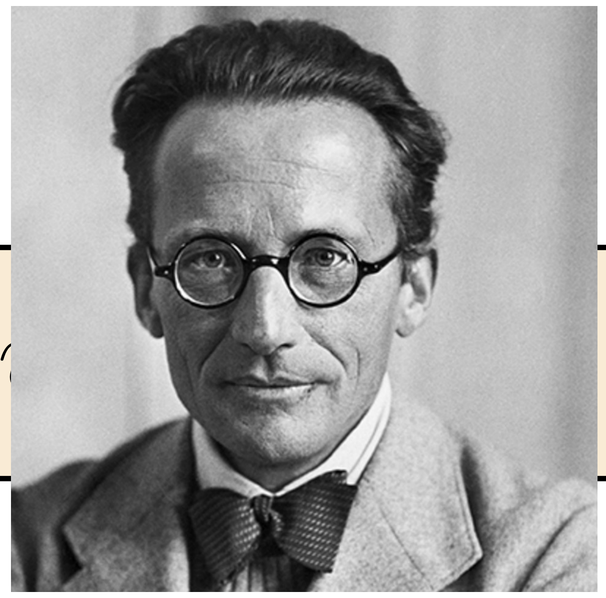
State Interpretation

M. Born

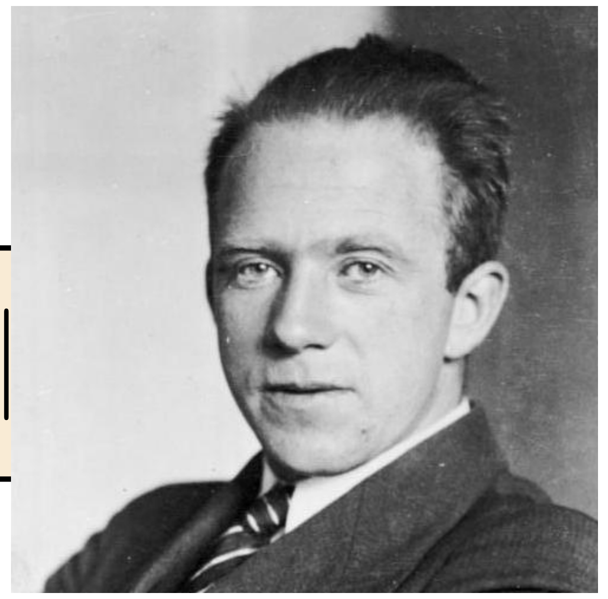


A. Einstein

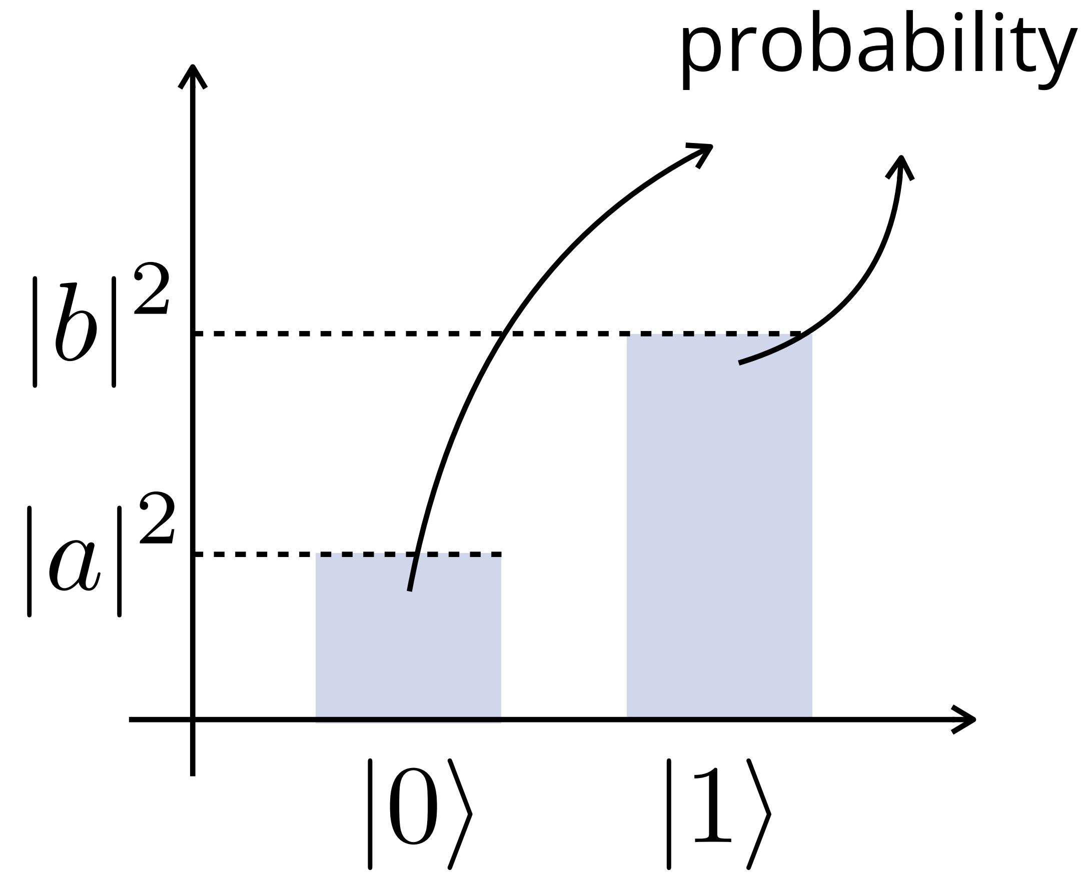
N. Bohr



E. Schrödinger



W. Heisenberg



Quantum State Interpretation	Probability Interpretation
A quantum state corresponds to a statistical **ensemble** of independent and identically prepared copies of a quantum system	**Frequentism**: The relative frequency of an event in the limit of sufficient many trials
A quantum state provides a complete description of an **individual** quantum system	**Bayesianism**: The degree of confidence of a hypothesis based on the prior knowledge

many worlds pilot waves Copenhagen interpretation quantum Bayesianism

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Composite States

Composite states: $|\psi\rangle^{(1)} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = c_0 |0\rangle + c_1 |1\rangle$ $\quad |\psi\rangle^{(2)} = \begin{bmatrix} d_0 \\ d_1 \end{bmatrix} = d_0 |0\rangle + d_1 |1\rangle$

$$|\Psi\rangle = |\psi\rangle^{(1)} \otimes |\psi\rangle^{(2)} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} \otimes \begin{bmatrix} d_0 \\ d_1 \end{bmatrix} = \begin{bmatrix} c_0 \begin{bmatrix} d_0 \\ d_1 \end{bmatrix} \\ c_1 \begin{bmatrix} d_0 \\ d_1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} c_0 d_0 \\ c_0 d_1 \\ c_1 d_0 \\ c_1 d_1 \end{bmatrix}$$

Can we always decompose a given vector into a tensor product of vectors?

$$\begin{bmatrix} \star \\ \star \end{bmatrix} \otimes \begin{bmatrix} \star \\ \star \end{bmatrix} \quad \overset{?}{\longleftarrow} \quad \begin{bmatrix} \star \\ \star \\ \star \\ \star \end{bmatrix}$$

Answer: No!

(Bell state) $|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

convention

$$|ij\rangle = |i\rangle \otimes |j\rangle$$

Entanglement

A composite quantum state that cannot be written as a tensor product of two smaller quantum states is called **entangled** state

$$|\Psi\rangle \neq |\psi\rangle^{(1)} \otimes |\psi\rangle^{(2)}$$

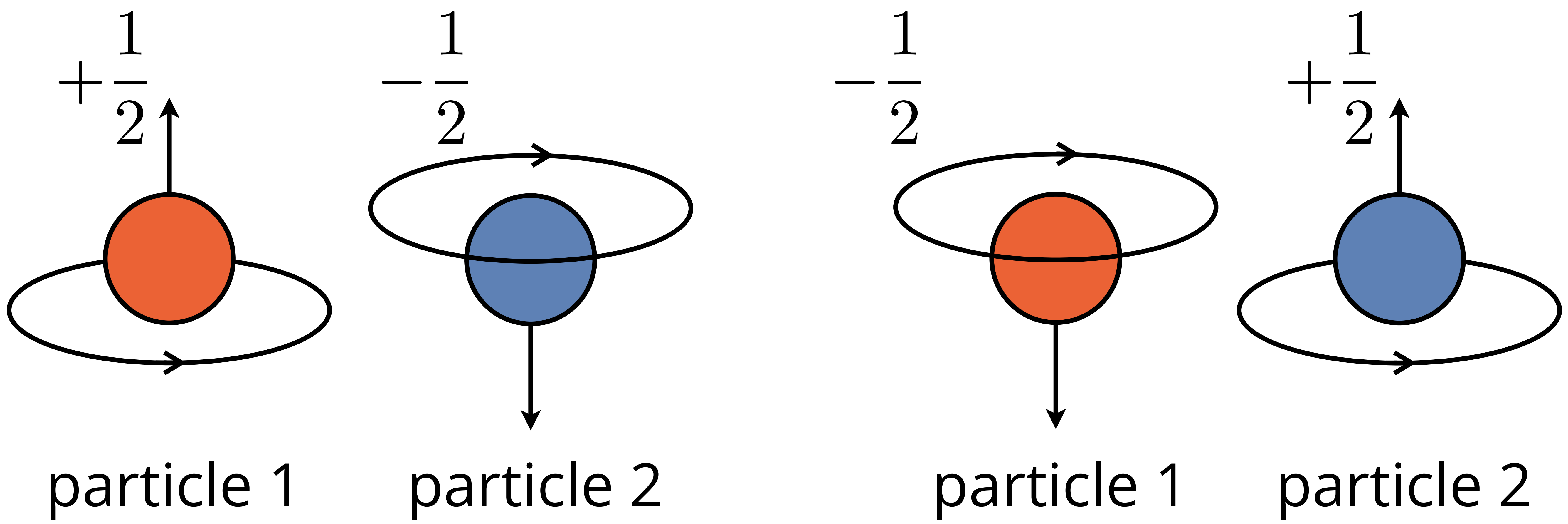
Otherwise it is called **separable** or **product** state.

- **Entangled systems share a common property**, but we don't know which part has which share until we measure it.
- For example, two particles have a total (sum) spin of zero, but we don't know the spin of each individual particle before we measure it.

$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$|0\rangle : +\frac{1}{2}$ spin

$|1\rangle : -\frac{1}{2}$ spin



What Entanglement is NOT

Far Apart, 2 Particles Respond Faster Than Light

Give this article

Can quantum entanglement send info faster than light? Yes.
Einstein was wrong on this one. The link between two electrons vibrating in unison does send information faster than light. But Einstein still has the last laugh, because...

7:48 AM · Nov 17, 2018 · Twitter Web App

Spooky! Quantum Action Is 10,000 Times Faster Than Light

BREAKING—Successful faster-than-speed-of-light demonstration of **QUANTUM TELEPORTATION** of up to 44 kilometer. @Caltech & @Fermilab scientists teleported quantum info for a sustained period across distance of 44 km via quantum entanglement teleportation.
inqnet.caltech.e

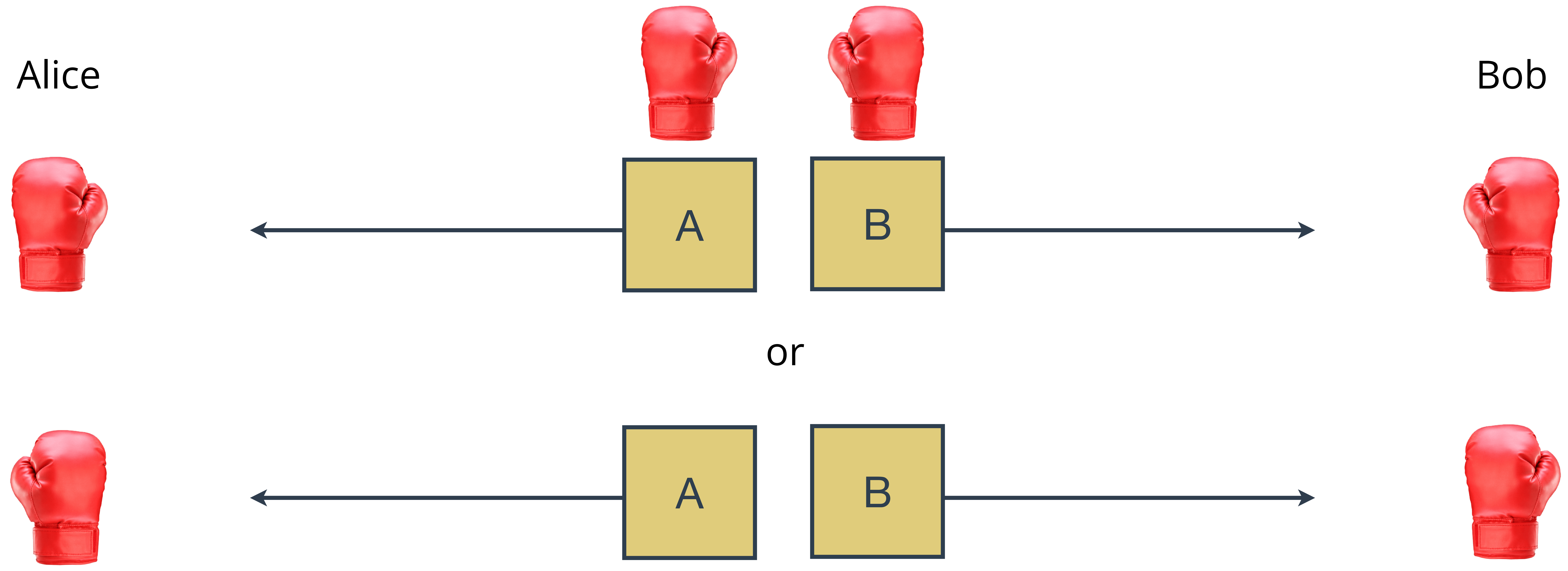
Quantum "spooky action at a distance" travels at least 10,000 times faster than light

Quantum weirdness wins again: Entanglement clocks in at 10,000+ times faster than light

NASA scientists achieve long-distance 'quantum teleportation' over 27 miles for the first time – paving the way for unhackable networks that transfer data faster than the speed of light

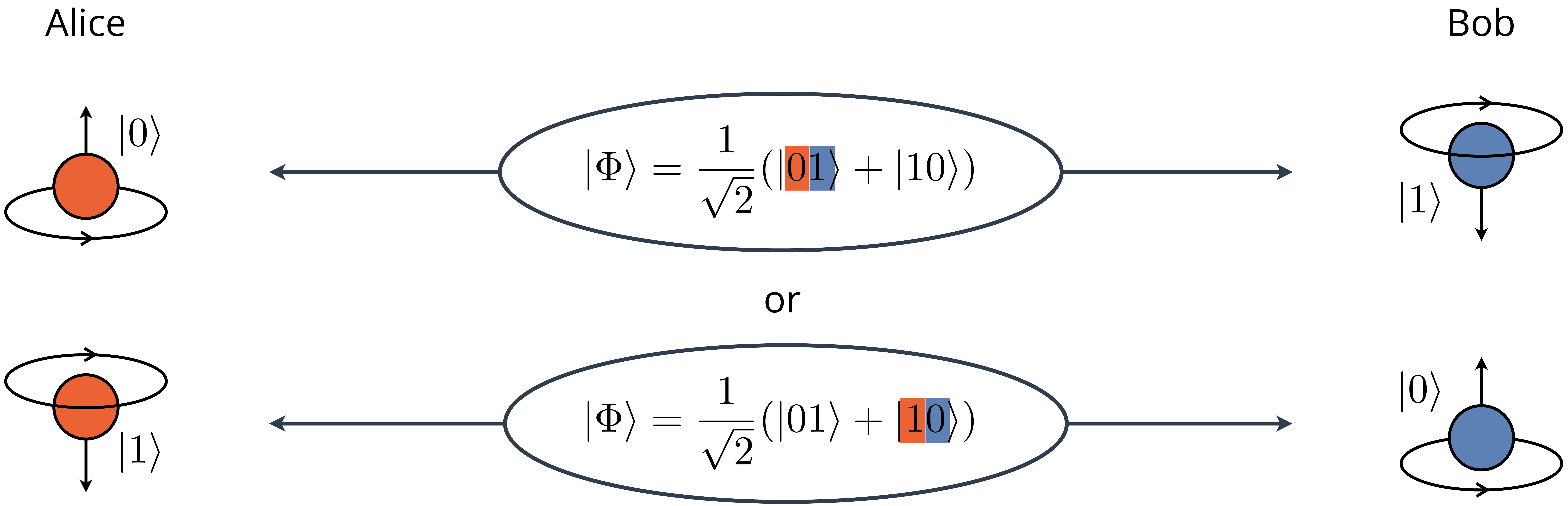
- Entanglement **DOES NOT** allow faster-than-light communication.
- Entanglement **DOES NOT** contain information. It contains correlations about information.

Classical Correlations



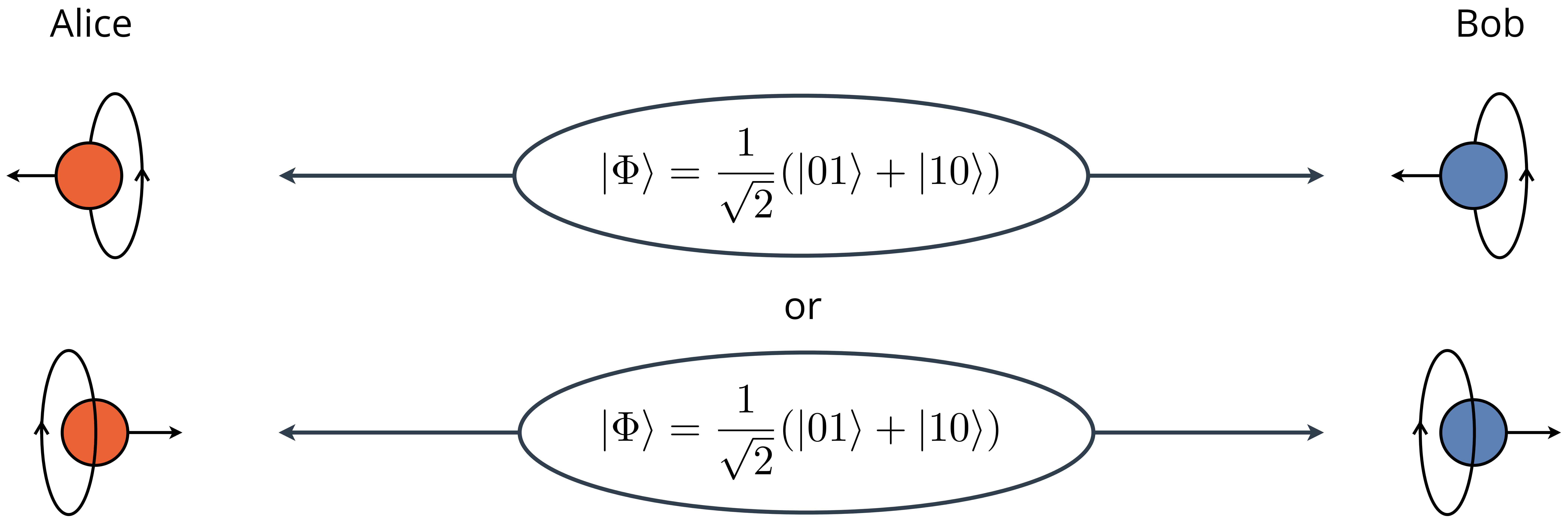
The gloves are perfectly correlated but no information has traveled from one place to another!

Quantum Correlations

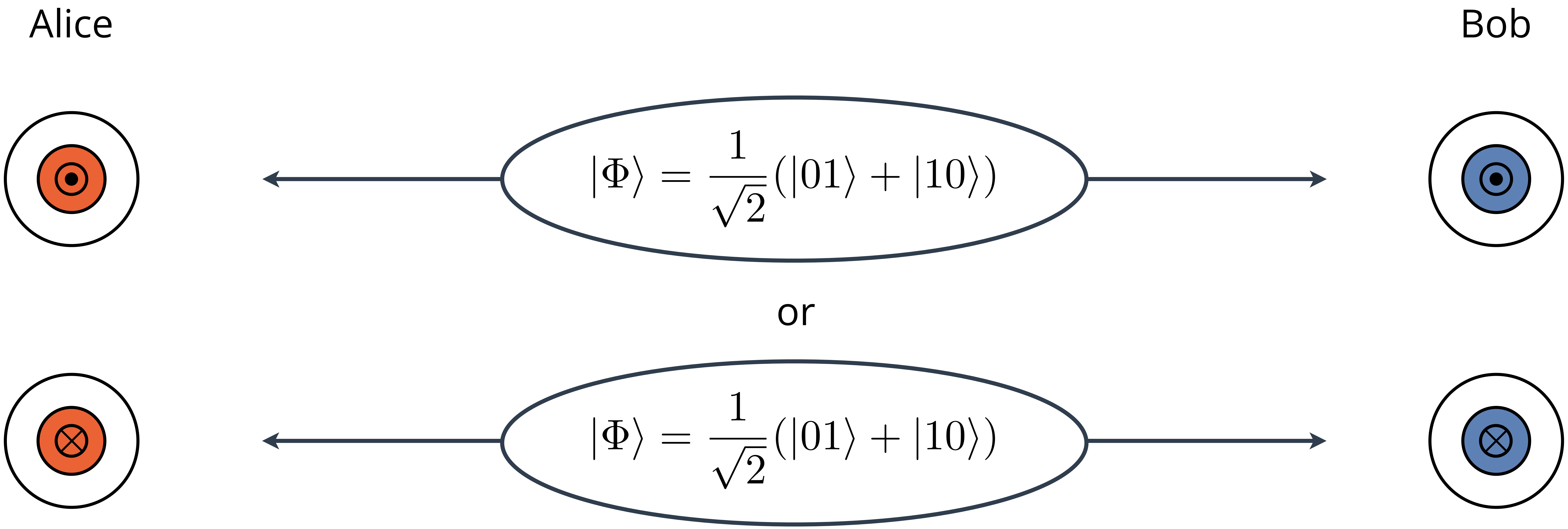


- The spins are perfectly correlated but no information has traveled from one place to another!
- So, what how are quantum correlations different from classical ones?

Quantum Correlations



Quantum Correlations

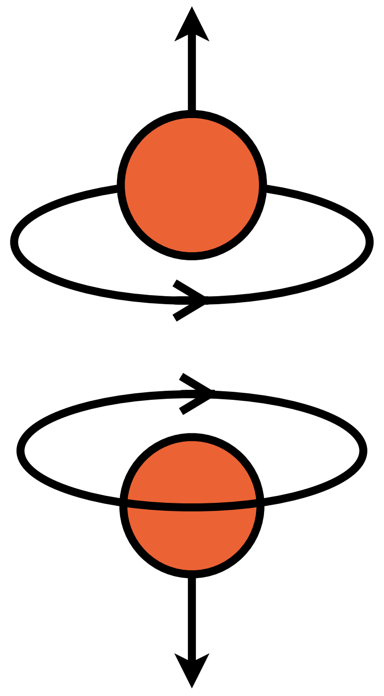


Quantum Correlations

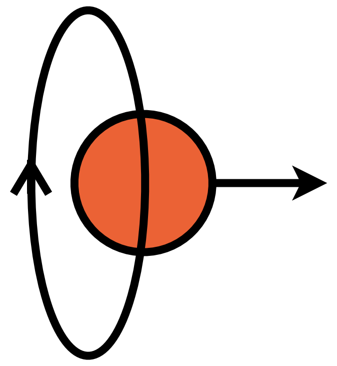
$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}(|{\uparrow\downarrow}\rangle + |{\downarrow\uparrow}\rangle)$$

$$|0\rangle = |{\uparrow}\rangle = \begin{bmatrix}1\\0\end{bmatrix}$$

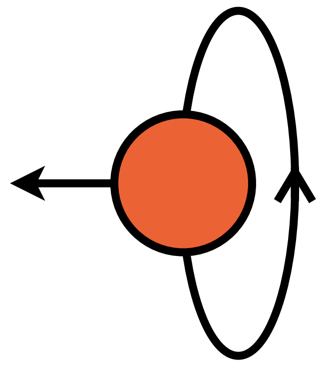
$$|1\rangle = |{\downarrow}\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$



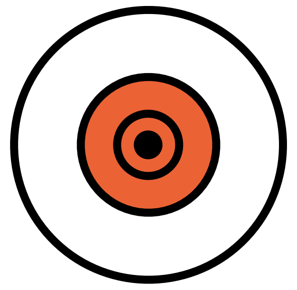
$$|{\rightarrow}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}(|{\uparrow}\rangle + |{\downarrow}\rangle)$$



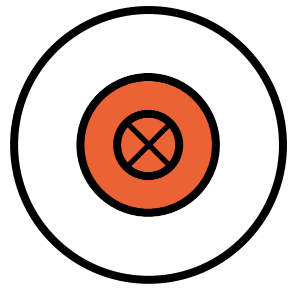
$$|{\leftarrow}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{\sqrt{2}}(|{\uparrow}\rangle - |{\downarrow}\rangle)$$



$$|{\odot}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\\imath\end{bmatrix} = \frac{1}{\sqrt{2}}(|{\uparrow}\rangle + \imath\,|{\downarrow}\rangle)$$



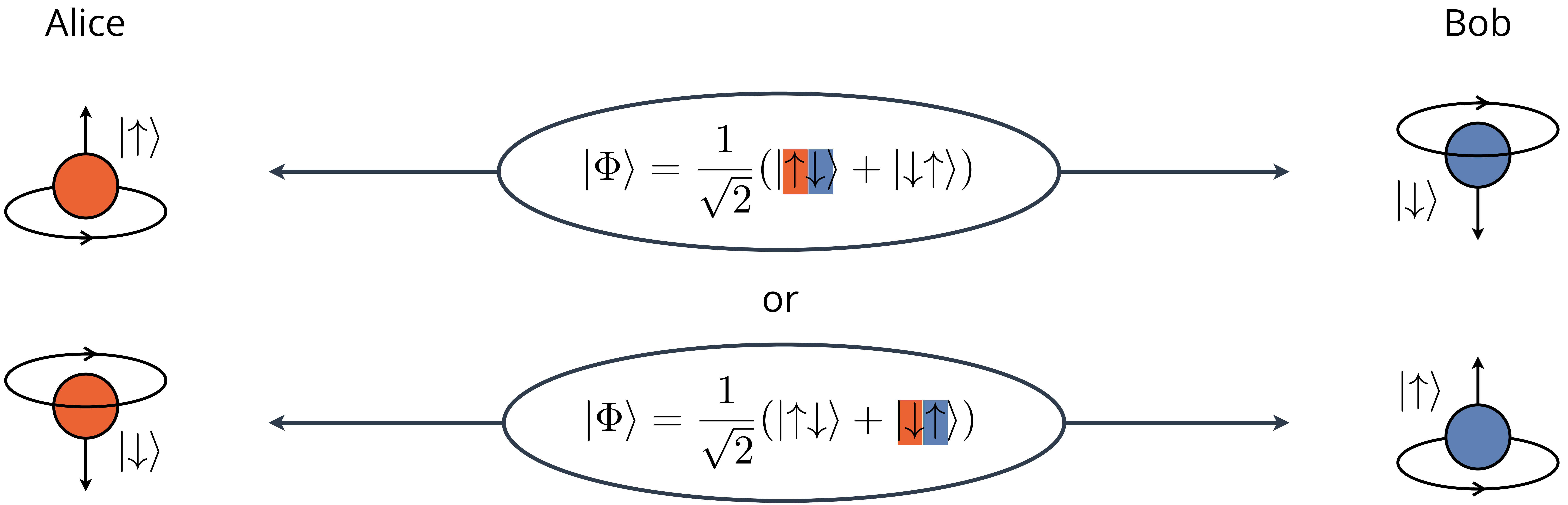
$$|{\otimes}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-\imath\end{bmatrix} = \frac{1}{\sqrt{2}}(|{\uparrow}\rangle - \imath\,|{\downarrow}\rangle)$$



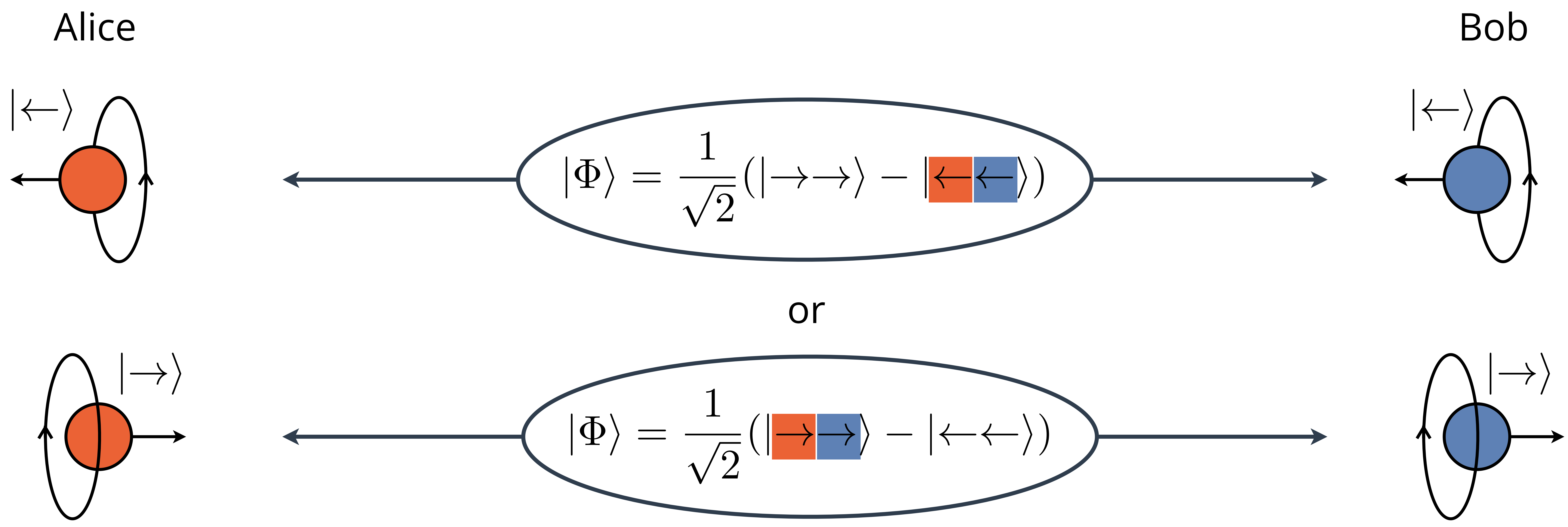
$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|{\rightarrow\rightarrow}\rangle - |{\leftarrow\leftarrow}\rangle)$$

$$|\Phi\rangle = \frac{-\imath}{\sqrt{2}}(|{\odot\odot}\rangle - |{\otimes\otimes}\rangle)$$

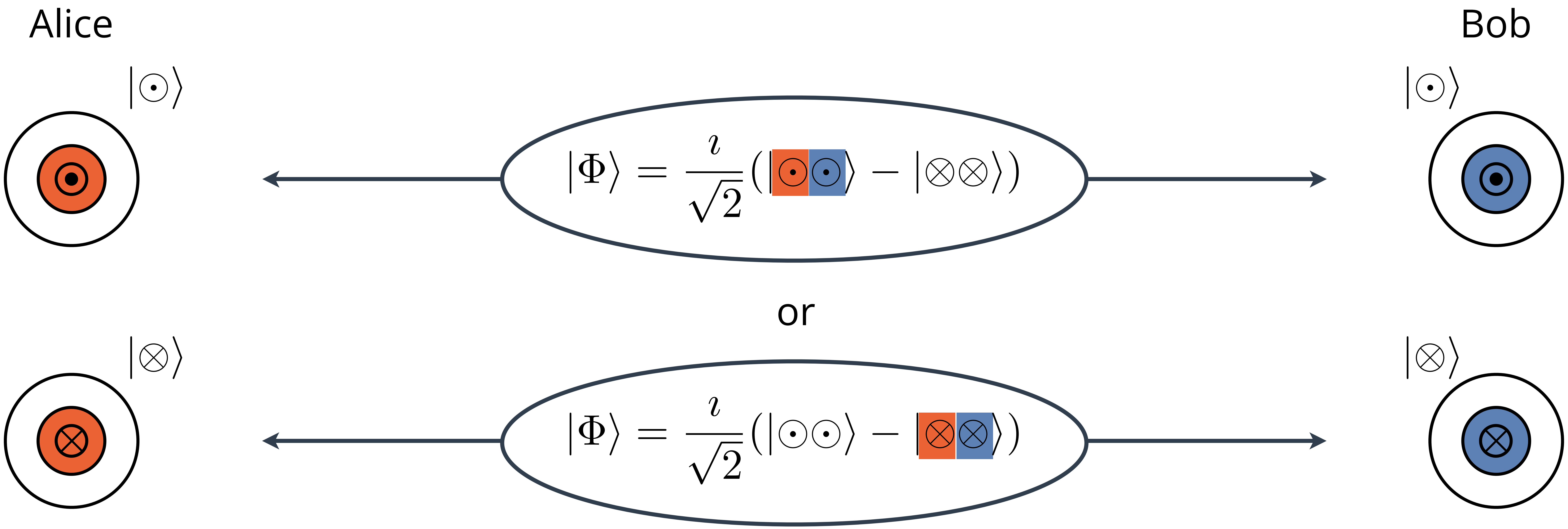
Quantum Correlations



Quantum Correlations



Quantum Correlations



- Quantum correlations are **stronger** than classical correlations
- In quantum systems **multiple properties can be simultaneously correlated**

Characterizing Entanglement

Schmidt decomposition: $|\Psi\rangle = \sum_i \sqrt{\lambda_i}|e_i\rangle \otimes |h_i\rangle$

$\{|e_i\rangle, |h_i\rangle\}$: orthonormal basis

$\{\lambda_0, \lambda_1\}$: Schmidt coefficients

- The number of non-zero Schmidt coefficients identifies entanglement

$\lambda_0 = 1$ & $\lambda_1 = 0 \quad \Rightarrow \quad |\Psi\rangle = |e_0\rangle \otimes |h_0\rangle$ (separable)

$\lambda_0 \neq 0$ & $\lambda_1 \neq 0 \quad \Rightarrow \quad |\Psi\rangle = \lambda_0|e_0\rangle \otimes |h_0\rangle + \lambda_1|e_1\rangle \otimes |h_1\rangle$ (entangled)

- Entanglement is the superposition of composite quantum systems

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad \lambda_0 = \lambda_1 = \frac{1}{\sqrt{2}} \qquad |e_0^{(1)}\rangle = |e_0^{(2)}\rangle = |0\rangle \qquad |e_1^{(1)}\rangle = |e_1^{(2)}\rangle = |1\rangle$$

Characterizing Entanglement

Schmidt decomposition: $|\Psi\rangle = \sum_i \sqrt{\lambda_i}|e_i\rangle \otimes |h_i\rangle$

$\{|e_i\rangle, |h_i\rangle\}$: orthonormal basis

$\{\lambda_0, \lambda_1\}$: Schmidt coefficients

- The number of non-zero Schmidt coefficients identifies entanglement

$\lambda_0 = 1$ & $\lambda_1 = 0 \Rightarrow |\Psi\rangle = |e_0\rangle \otimes |h_0\rangle$ (separable)

$\lambda_0 \neq 0$ & $\lambda_1 \neq 0 \Rightarrow |\Psi\rangle = \lambda_0|e_0\rangle \otimes |h_0\rangle + \lambda_1|e_1\rangle \otimes |h_1\rangle$ (entangled)

- Entanglement is the superposition of composite quantum systems

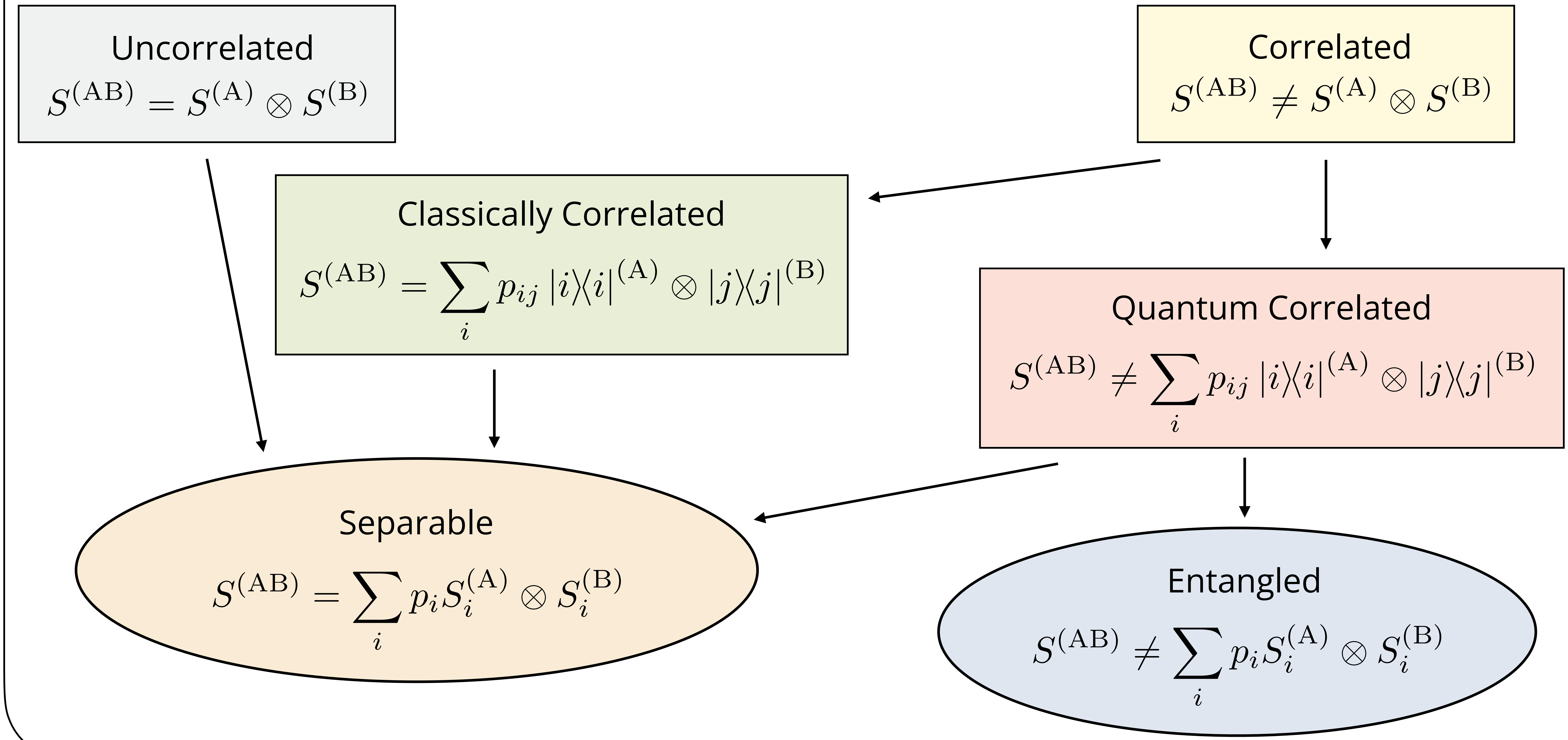
$|\Psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

$\lambda_0 = 1$ & $\lambda_1 = 0$

$|e_0^{(1)}\rangle = |e_0^{(2)}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$|e_1^{(1)}\rangle = |e_1^{(2)}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Entanglement in Mixed States



Detecting Entanglement

Assuming **complete knowledge of the quantum state** S, entanglement detection reduces to the **verification of a mathematical condition**.

(Peres-Horodecki Criterion) Let us have a quantum state S and a positive map $\mathcal{M} : \mathcal{M}(S) \to S'$
For separable states the map $[\mathcal{M} \otimes \mathbb{1}](S)$ must yield a positive operator.

$$\mathcal{M} = \mathcal{T} : |i\rangle\langle j| \to |j\rangle\langle i| \qquad \Phi = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$[\mathcal{T} \otimes \mathbb{1}_2]\,(\Phi) = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2} \qquad \lambda_4 = -\frac{1}{2}$$

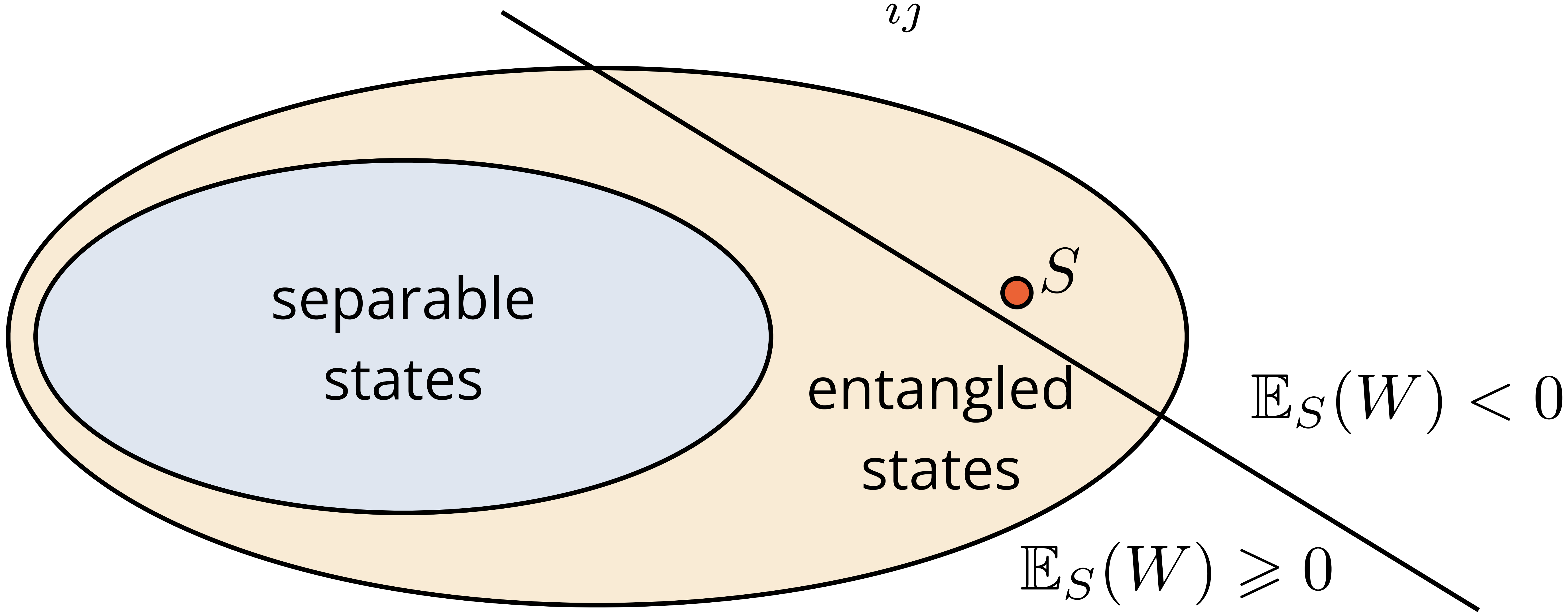
A. Peres Phys. Rev. Lett. 77, 1413 (1996)

Detecting Entanglement

Assuming **no prior knowledge of the quantum state** S, entanglement detection can be achieved using specially designed measurements, known as **entanglement witnesses**.

An entanglement witness is a Hermitian operator that yields a non-negative mean value with respect to any separable state. Thus, the detection of a negative value implies entanglement

$$\mathbb{E}_S(W) < 0 \Rightarrow S: \text{ entangled} \qquad W = \sum_{ij} c_{ij} A_i \otimes B_j \qquad c_{ij} \in \mathbb{R}$$



O. Guhne et al. Mod. Opt. 50, 1079 (2003)

Detecting Entanglement

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$$\Phi = \frac{1}{2}\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1\end{bmatrix}$$

$$\mathbb{E}_\Phi(W) = \mathrm{tr}(\Phi W) = 2 - 2\sqrt{2} < 0$$

$$W = 2\mathbb{1}_4 - C$$

$$C = A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2$$

$$A_1 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \qquad A_2 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$B_1 = (A_1 + A_2)/\sqrt{2} \qquad B_2 = (A_2 - A_1)/\sqrt{2}$$

O. Guhne et al. Mod. Opt. 50, 1079 (2003)

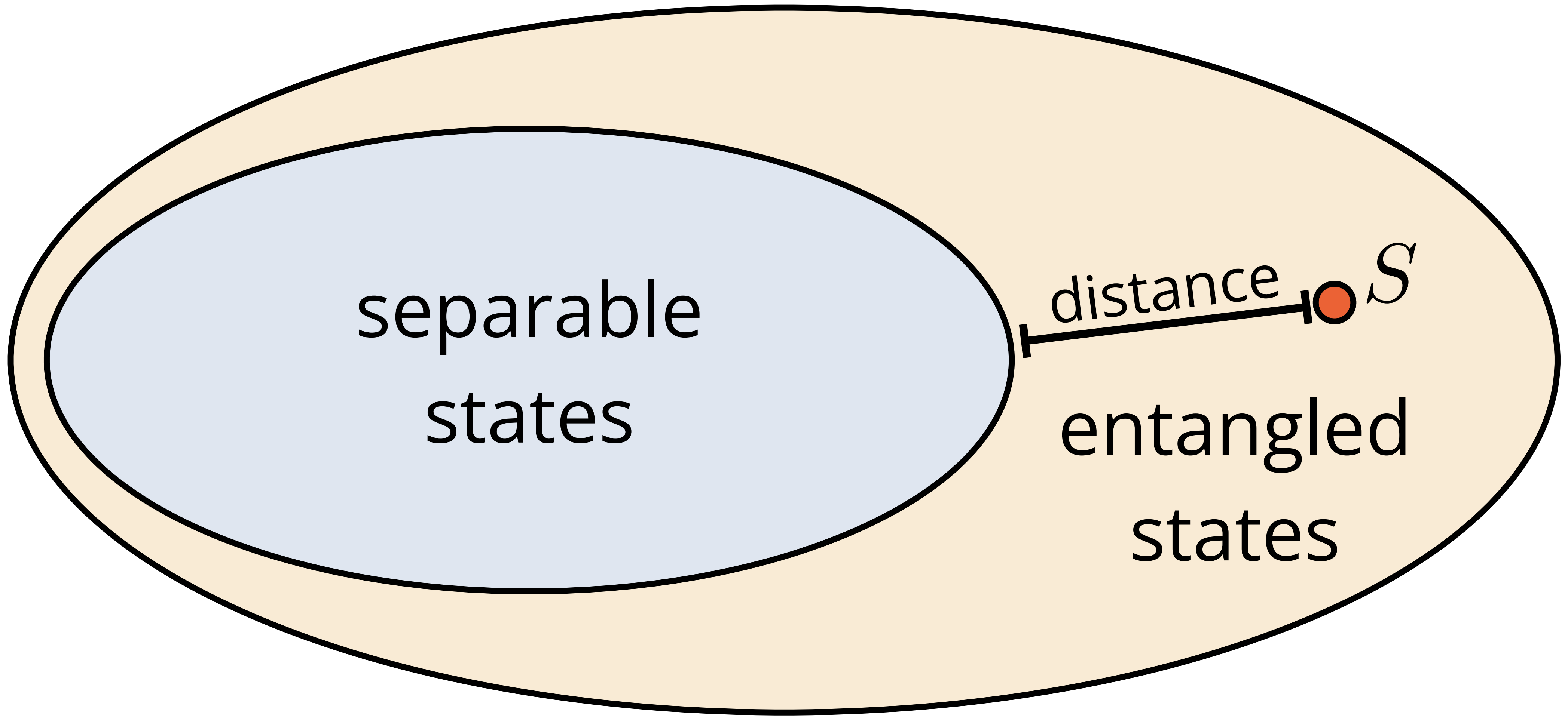
Quantifying Entanglement

$[\mathcal{T} \otimes \mathbb{1}](S) \leqslant 0$

detects entanglement

λ_i : eigenvalues of $[\mathcal{T} \otimes \mathbb{1}](S)$

Negativity $\mathrm{E}_N(S) := \left| \sum_{\lambda_i < 0} \lambda_i \right|$



What is the **minimum distance** between an entangled state and the set of separable states?

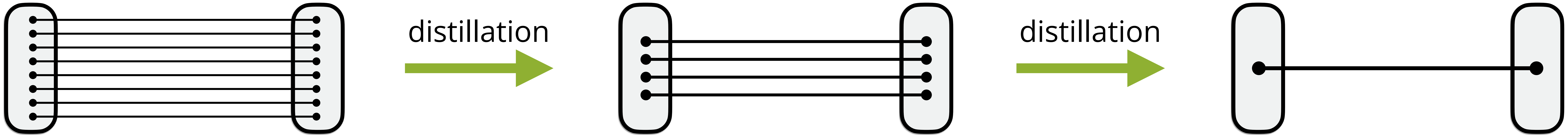
Relative entropy of entanglement $\mathrm{E}_R(S) = \min_{S_{\mathrm{sep}}} \mathbb{H}(S||S_{\mathrm{sep}})$

$\mathbb{H}(X||Y) := \mathrm{tr}(X \log X) - \mathrm{tr}(X \log Y)$

G. Vidal and R. F. Werner Phys. Rev. A 65, 032314 (2002)

V. Vedral et al. Rev. Lett. 78, 2275–2279 (1997)

Quantifying Entanglement



- What is the **maximum** amount of Bell states we can **distill** from a state?

$$S^{\otimes m} \xrightarrow{\text{LOCC}} |\Phi\rangle^{\otimes n} \qquad \mathrm{E}_D := \max_{\text{LOCC}} \left\{ \frac{n}{m} \right\}$$

Distillable entanglement $\quad \mathrm{E}_D(S) := \sup_r \left\{ \lim_{n\to\infty} \left[\inf_{\Lambda_i} \left\| \Lambda_i(S^{\otimes n}) - (|\Phi\rangle\langle\Phi|)^{rn} \right\|_1 \right] = 0 \right\}$

- What is the **minimum** amount of Bell states **used** to create a state?

$$|\Phi\rangle^{\otimes n} \xrightarrow{\text{LOCC}} S^{\otimes m} \qquad \mathrm{E}_C := \min_{\text{LOCC}} \left\{ \frac{n}{m} \right\}$$

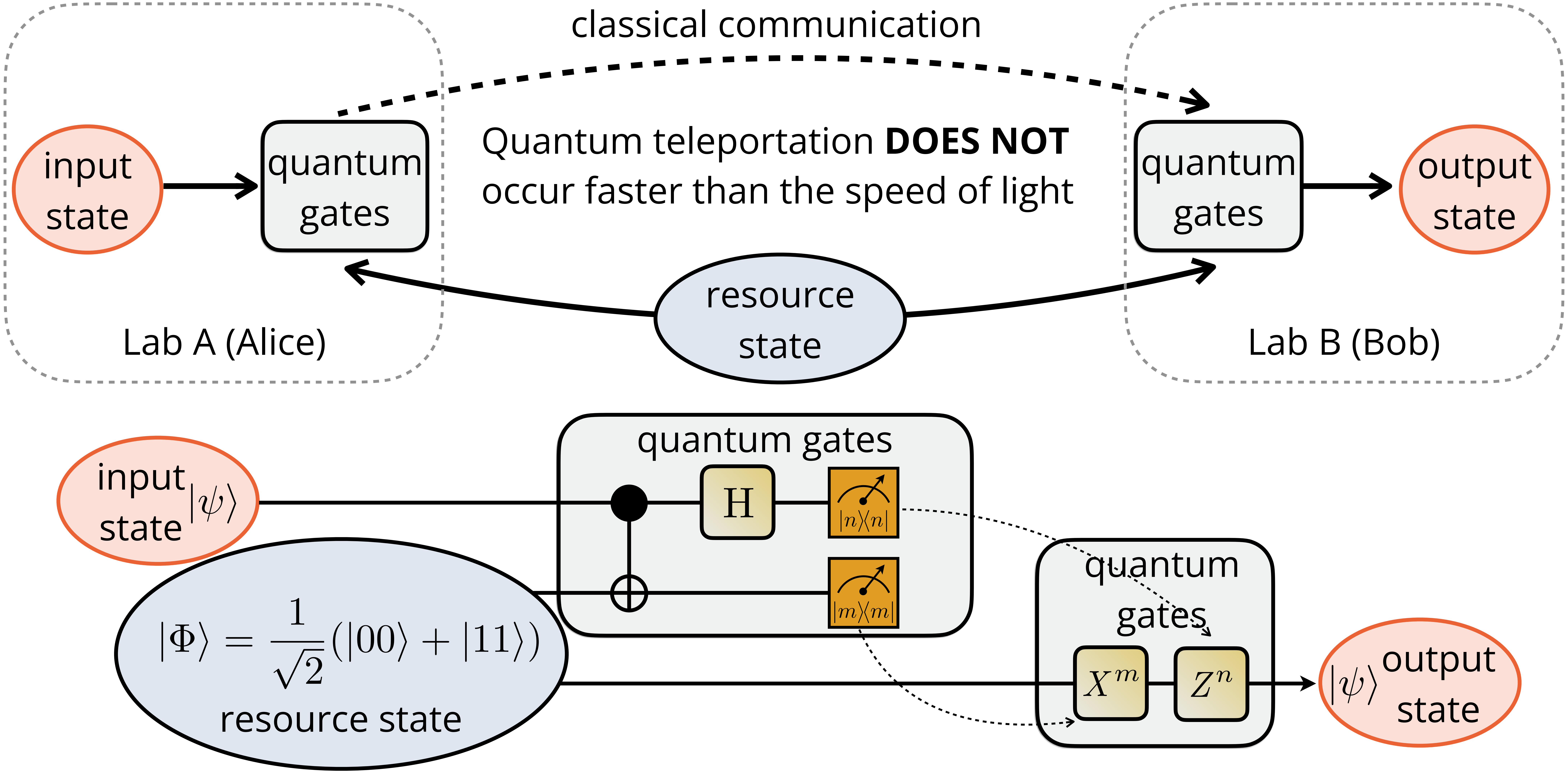
Entanglement cost $\quad \mathrm{E}_C(S) := \inf_r \left\{ \lim_{n\to\infty} \left[\inf_{\Lambda_i} \left\| S^{\otimes n} - \Lambda_i \left[(|\Phi\rangle\langle\Phi|)^{rn} \right] \right\|_1 \right] = 0 \right\}$

C. H. Bennett et al. Phys. Rev. Lett. 76, 722–725 (1996)

Outline of the Presentation

- **Introduction**
  - Classical properties and states
  - Quantum properties and states
- **Entanglement**
  - Entanglement definition
  - Entanglement detection
  - Entanglement quantification
- **Applications**
  - Teleportation

Quantum Teleportation

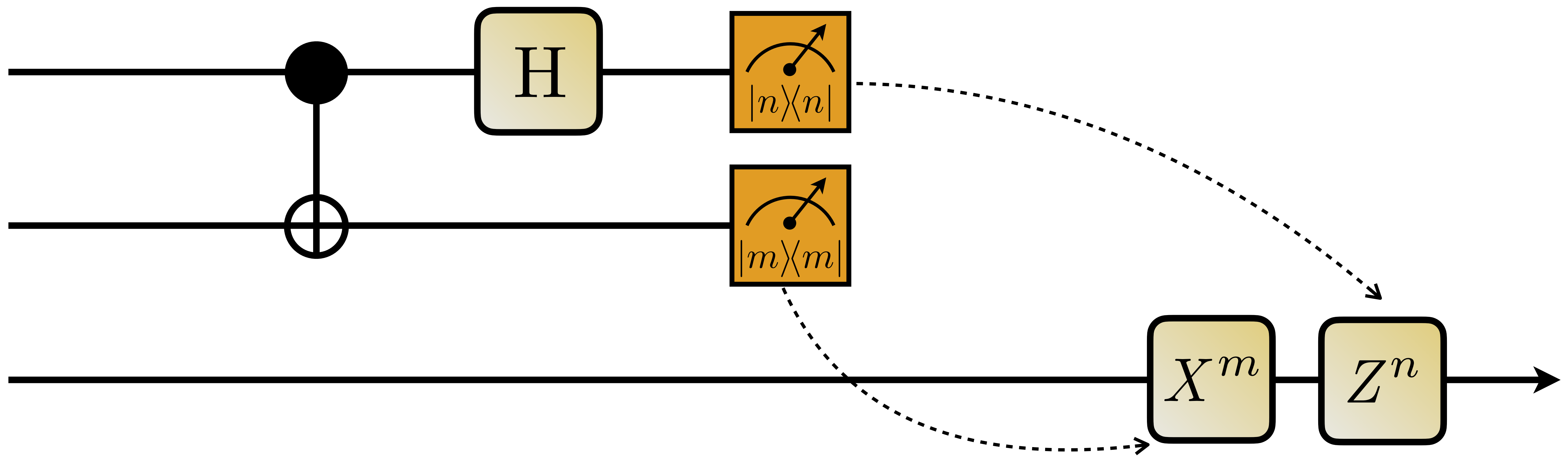


C. H. Bennett, et al., Phys. Rev. Lett. 70, 1895 (1993)

Quantum Teleportation

$$|\psi\rangle_{\text{in}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|\Psi\rangle = \sqrt{1-x}\,|00\rangle + \sqrt{x}\,|11\rangle$$



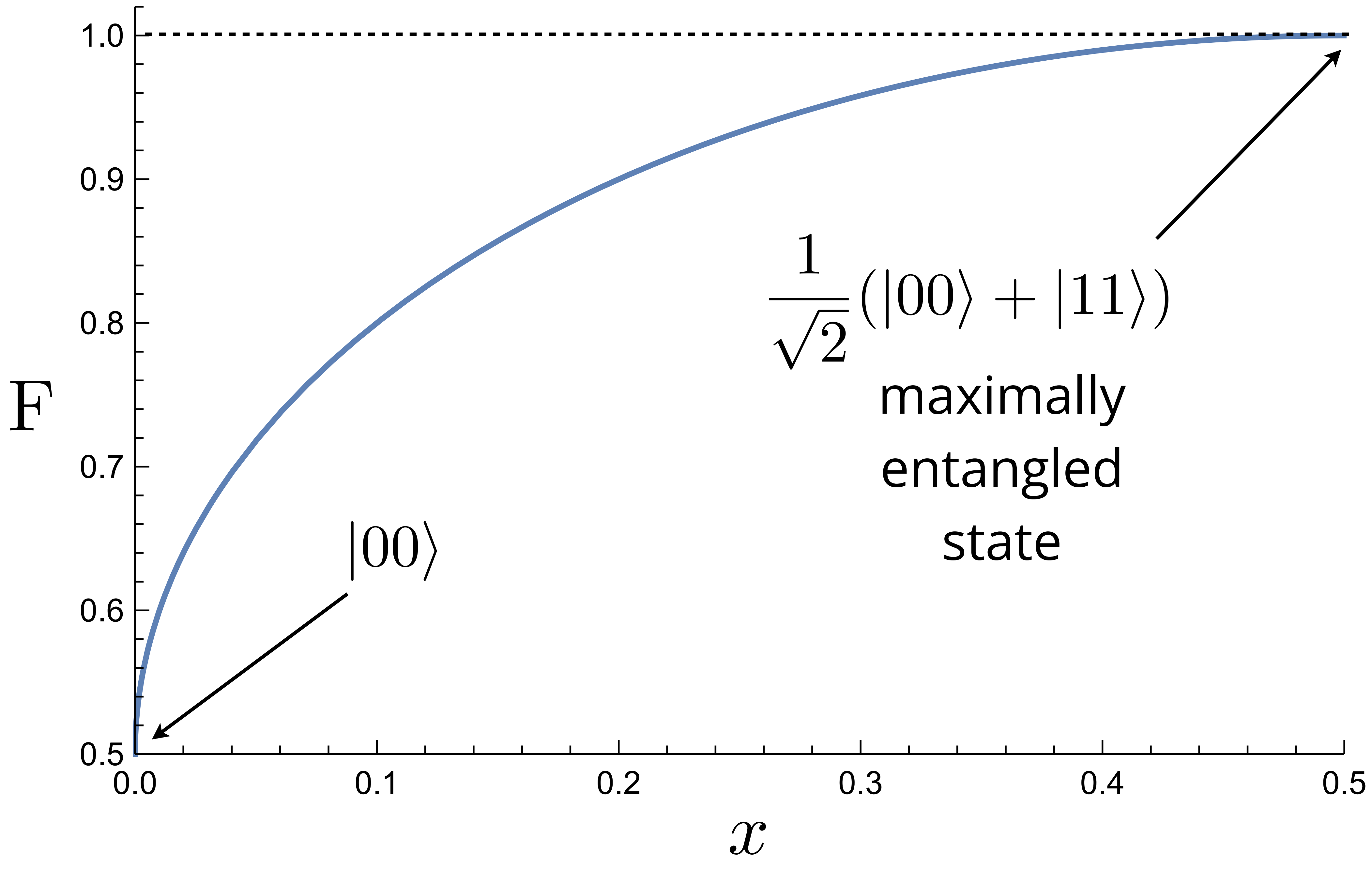
$$|\psi\rangle_{\text{out}}$$

$x = 0 \Rightarrow |\Psi\rangle = |00\rangle$ (separable)

$x = \frac{1}{2} \Rightarrow |\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ (entangled)

Fidelity: $\mathrm{F} = |\langle\psi_{\text{in}}|\psi_{\text{out}}\rangle|^2 \quad 0 \leqslant \mathrm{F} \leqslant 1$

$$\mathrm{F} = 1 \Leftrightarrow |\psi_{\text{in}}\rangle = |\psi_{\text{out}}\rangle$$



Conclusion

- Entanglement is a fundamental physical property
- Entanglement is used as a resource in quantum technology applications
- Characterization of entanglement is an interesting mathematical problem

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