Quantum Systems, Information, and Entanglement

Lecture 2. Teleportation and Entanglement

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HARVARD UNIVERSITY

NARANGLAB
MATERIALS THEORY AT HARVARD

Course Outline

- **Lecture 1 — Introduction to Quantum Systems** (April 13, 2022)
- **Lecture 2 — Teleportation and Entanglement** (April 20, 2022)
- **Lecture 3 — Decoherence and Quantum Networks** (April 27, 2022)

Lecture 2 — Teleportation and Entanglement

- Quantum Gates
  - Single-qubit gates
  - Two-qubit gates
- Quantum Teleportation
  - Step-by-step analysis
- Entanglement
  - What entanglement is and what it is not
  - Entanglement as resource for teleportation

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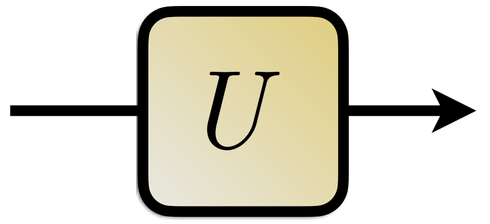
Quantum Information

- Information in quantum systems is encoded into **qubits** $|\psi\rangle = a|0\rangle + b|1\rangle$

- Information is measured through **measurements** $\Pi_0 = |0\rangle\langle 0| \rightarrow |0\rangle$ with probability $|a|^2$

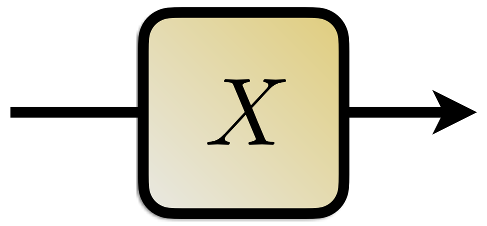
  $\Pi_1 = |1\rangle\langle 1| \rightarrow |1\rangle$ with probability $|b|^2$

- Information is processed through **quantum gates**



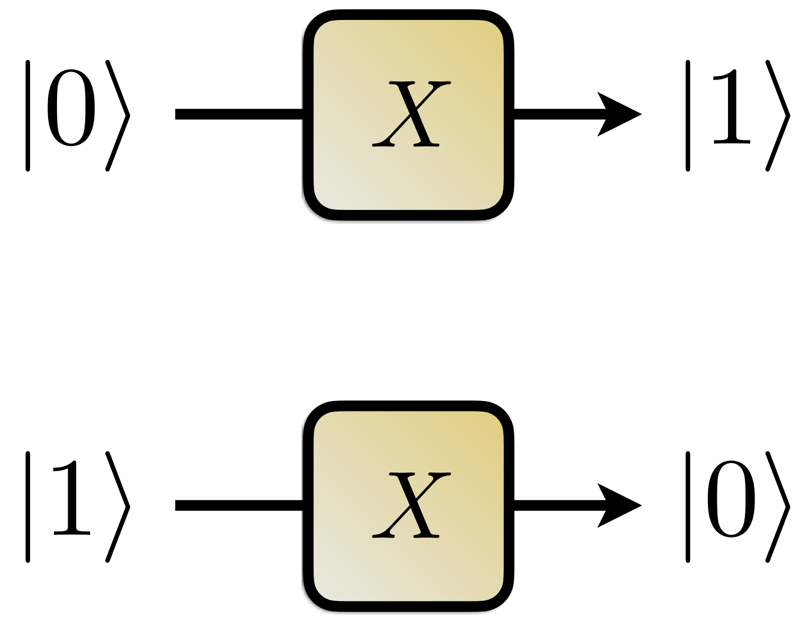
$$UU^\dagger = U^\dagger U = \mathbb{1}$$

$a|0\rangle + b|1\rangle$

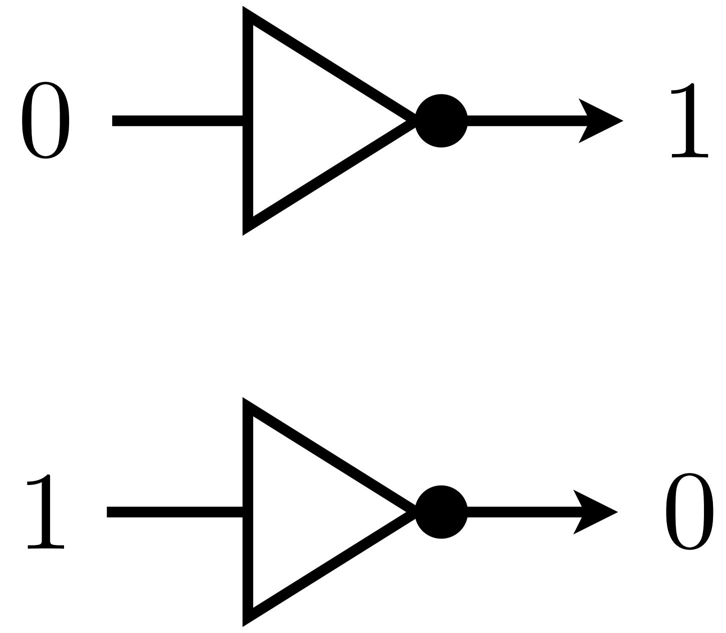


$b|0\rangle + a|1\rangle$

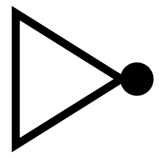
$$X = |0\rangle\langle 1| + |1\rangle\langle 0|$$



$$\mathrm{NOT} = X$$



$\mathrm{NOT} =$



Quantum Gates

Gate	Matrix	Operation		Name
$\rightarrow\boxed{X}\rightarrow$	$\|0\rangle\langle 1\| + \|1\rangle\langle 0\|$	$\|0\rangle \rightarrow\boxed{X}\rightarrow \|1\rangle$	$\|1\rangle \rightarrow\boxed{X}\rightarrow \|0\rangle$	Pauli Matrix X (NOT)
$\rightarrow\boxed{Y}\rightarrow$	$-\imath(\|0\rangle\langle 1\| - \|1\rangle\langle 0\|)$	$\|0\rangle \rightarrow\boxed{Y}\rightarrow \imath\,\|1\rangle$	$\|1\rangle \rightarrow\boxed{Y}\rightarrow -\imath\,\|0\rangle$	Pauli Matrix Y
$\rightarrow\boxed{Z}\rightarrow$	$\|0\rangle\langle 0\| - \|1\rangle\langle 1\|$	$\|0\rangle \rightarrow\boxed{Z}\rightarrow \|0\rangle$	$\|1\rangle \rightarrow\boxed{Z}\rightarrow -\|1\rangle$	Pauli Matrix Z
$\rightarrow\boxed{\mathrm{H}}\rightarrow$	$\frac{1}{\sqrt{2}}(X + Z)$	$\|0\rangle \rightarrow\boxed{\mathrm{H}}\rightarrow \frac{\|0\rangle + \|1\rangle}{\sqrt{2}}$	$\|1\rangle \rightarrow\boxed{\mathrm{H}}\rightarrow \frac{\|0\rangle - \|1\rangle}{\sqrt{2}}$	Hadamard
$\rightarrow\boxed{\varphi}\rightarrow$	$\|0\rangle\langle 0\| + e^{\imath\varphi}\,\|1\rangle\langle 1\|$	$\|0\rangle \rightarrow\boxed{\varphi}\rightarrow \|0\rangle$	$\|1\rangle \rightarrow\boxed{\varphi}\rightarrow e^{\imath\varphi}\,\|1\rangle$	Phase Shift

Multiple Qubits

Let us have two qubits

$$|\psi\rangle^{(1)} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = c_0 |0\rangle + c_1 |1\rangle \qquad |\psi\rangle^{(2)} = \begin{bmatrix} d_0 \\ d_1 \end{bmatrix} = d_0 |0\rangle + d_1 |1\rangle$$

The combined state is given by

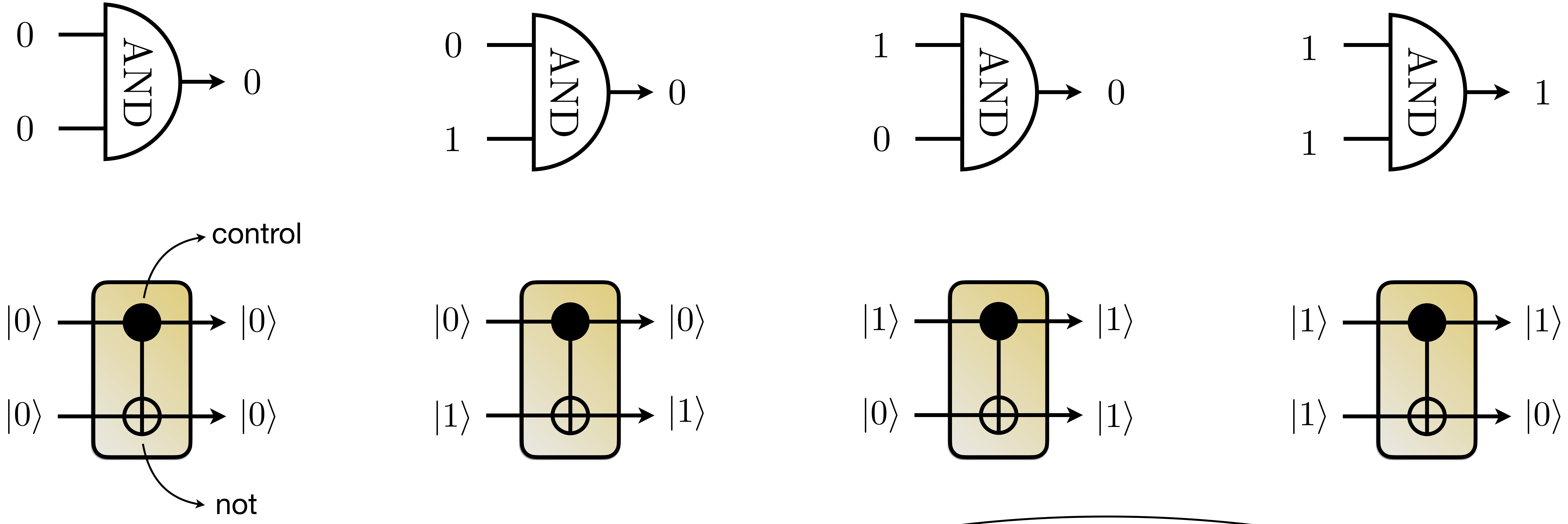
tensor product

$$|\Psi\rangle = |\psi\rangle^{(1)} \otimes |\psi\rangle^{(2)} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} \otimes \begin{bmatrix} d_0 \\ d_1 \end{bmatrix} = \begin{bmatrix} c_0 \begin{bmatrix} d_0 \\ d_1 \end{bmatrix} \\ c_1 \begin{bmatrix} d_0 \\ d_1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} c_0 d_0 \\ c_0 d_1 \\ c_1 d_0 \\ c_1 d_1 \end{bmatrix}$$

In Dirac notation we have

$$\begin{aligned} |\Psi\rangle = |\psi\rangle^{(1)} \otimes |\psi\rangle^{(2)} &= (c_0 |0\rangle + c_1 |1\rangle) \otimes (d_0 |0\rangle + d_1 |1\rangle) \\ &= c_0 d_0 |0\rangle \otimes |0\rangle + c_0 d_1 |0\rangle \otimes |1\rangle + c_1 d_0 |1\rangle \otimes |0\rangle + c_1 d_1 |1\rangle \otimes |1\rangle \\ &= c_0 d_0 |00\rangle + c_0 d_1 |01\rangle + c_1 d_0 |10\rangle + c_1 d_1 |11\rangle \end{aligned}$$

Two-Qubit Gates



$$\text{CNOT}\,|\psi\rangle = \text{CNOT}(a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle) = a|00\rangle + b|01\rangle + c|11\rangle + d|10\rangle$$

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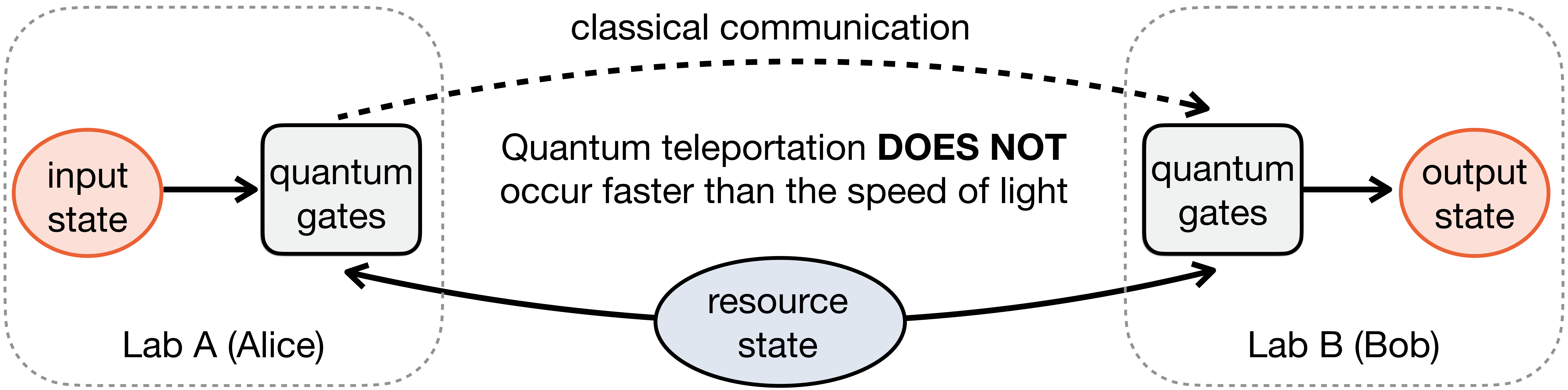
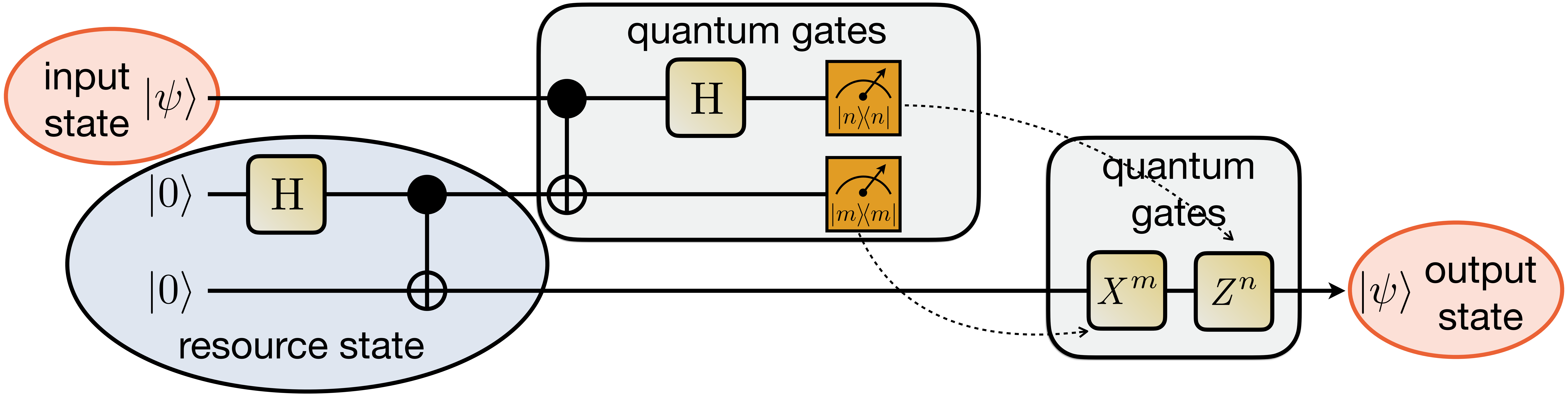
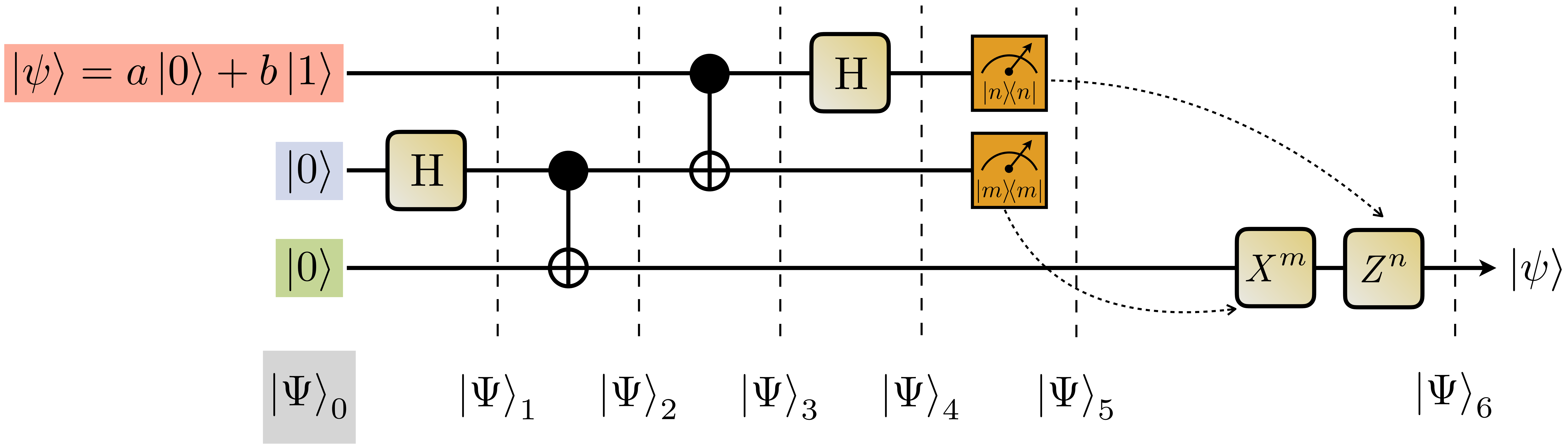
Quantum Teleportation





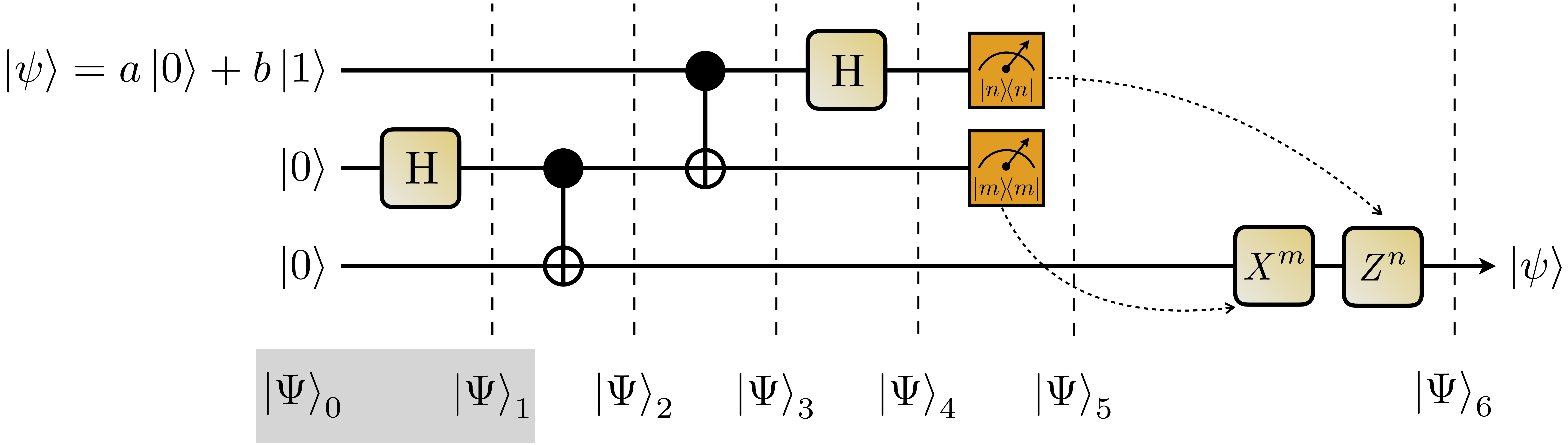
Charles H. Bennett, et al. Phys. Rev. Lett. 70, 1895 (1993)

Quantum Teleportation — Step 0



$$|\Psi\rangle_0 = |\psi\rangle \otimes |0\rangle \otimes |0\rangle = (a|0\rangle + b|1\rangle) \otimes |0\rangle \otimes |0\rangle = a|000\rangle + b|100\rangle$$

Quantum Teleportation — Step 1



$$|\Psi\rangle_1 = (\mathbb{1}_2 \otimes \mathrm{H} \otimes \mathbb{1}_2)\,|\Psi\rangle_0 = (\mathbb{1}_2 \otimes \mathrm{H} \otimes \mathbb{1}_2)(a\,|000\rangle + b\,|100\rangle)$$

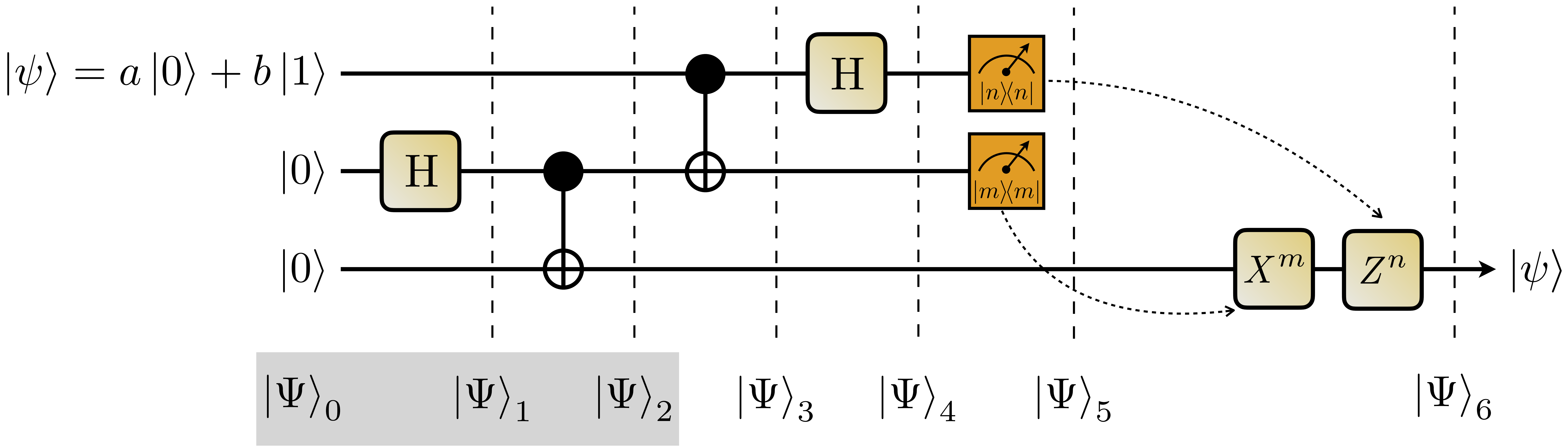
$$= a\,|0\rangle \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes |0\rangle + b\,|1\rangle \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2}}\left(a\,|000\rangle + a\,|010\rangle + b\,|100\rangle + b\,|110\rangle\right)$$

$$\mathrm{H}\,|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$\mathrm{H}\,|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Quantum Teleportation — Step 2

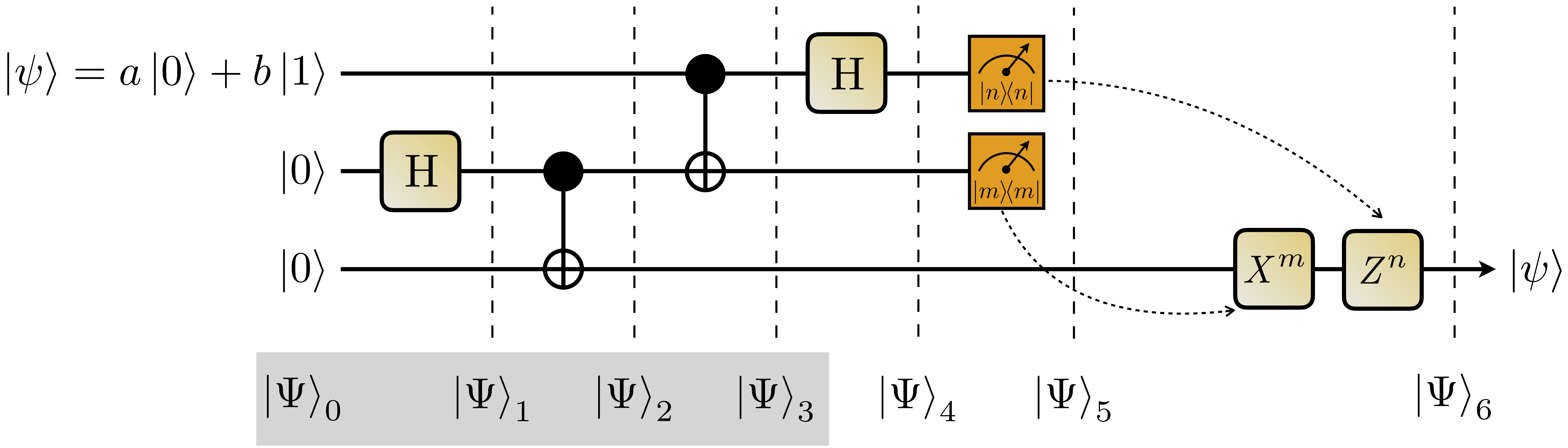


$$|\Psi\rangle_2 = (\mathbb{1}_2 \otimes \text{CNOT})\,|\Psi\rangle_1 = (\mathbb{1}_2 \otimes \text{CNOT})\left[\frac{1}{\sqrt{2}}\left(a\,|000\rangle + a\,|010\rangle + b\,|100\rangle + b\,|110\rangle\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left(a\,|000\rangle + a\,|011\rangle + b\,|100\rangle + b\,|111\rangle\right)$$

$$\text{CNOT}\,|00\rangle = |00\rangle$$
$$\text{CNOT}\,|01\rangle = |01\rangle$$
$$\text{CNOT}\,|10\rangle = |11\rangle$$
$$\text{CNOT}\,|11\rangle = |10\rangle$$

Quantum Teleportation — Step 3

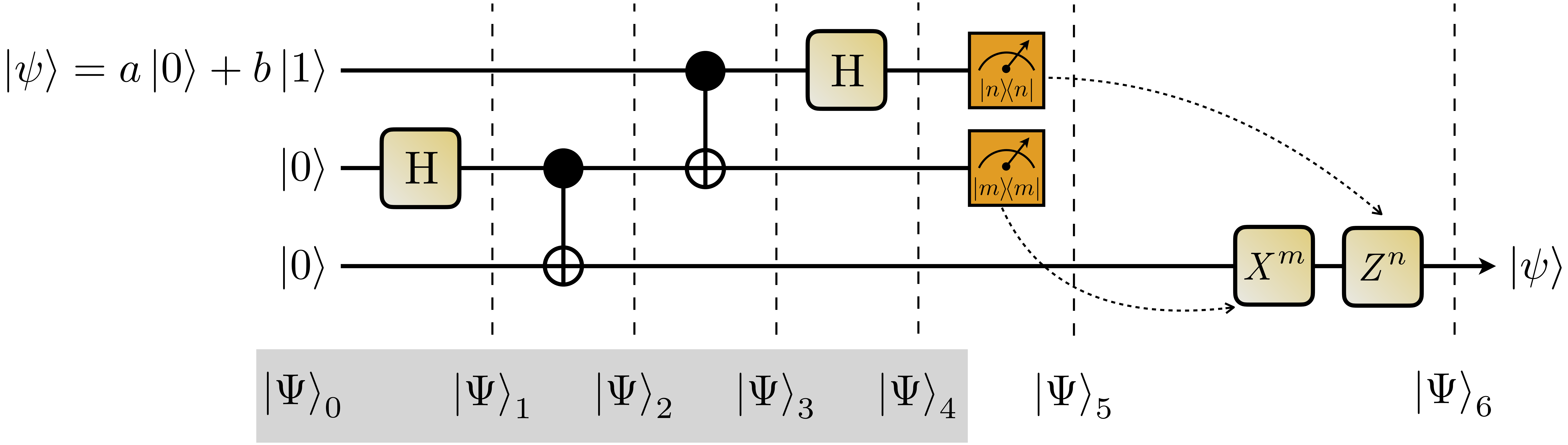


$$|\Psi\rangle_3 = (\text{CNOT} \otimes \mathbb{1}_2)|\Psi\rangle_2 = (\text{CNOT} \otimes \mathbb{1}_2)\left[\frac{1}{\sqrt{2}}\left(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left(a|000\rangle + a|011\rangle + b|110\rangle + b|101\rangle\right)$$

$$\text{CNOT}|00\rangle = |00\rangle$$
$$\text{CNOT}|01\rangle = |01\rangle$$
$$\text{CNOT}|10\rangle = |11\rangle$$
$$\text{CNOT}|11\rangle = |10\rangle$$

Quantum Teleportation — Step 4



$$|\Psi\rangle_4 = (\mathrm{H} \otimes \mathbb{1}_2 \otimes \mathbb{1}_2)\,|\Psi\rangle_3 = (\mathrm{H} \otimes \mathbb{1}_2 \otimes \mathbb{1}_2)\left[\frac{1}{\sqrt{2}}\left(a\,|000\rangle + a\,|011\rangle + b\,|110\rangle + b\,|101\rangle\right)\right]$$

$$= \frac{1}{2}\big[a(|0\rangle + |1\rangle) \otimes |00\rangle + a(|0\rangle + |1\rangle) \otimes |11\rangle$$

$$+ b(|0\rangle - |1\rangle) \otimes |10\rangle + b(|0\rangle - |1\rangle) \otimes |01\rangle\big]$$

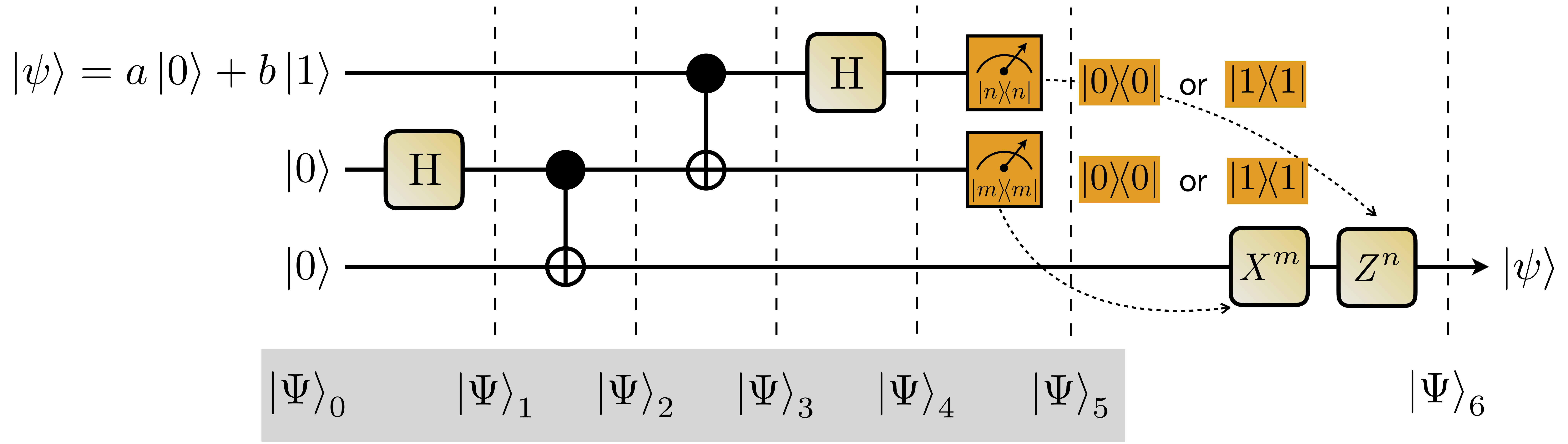
$$\mathrm{H}\,|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$\mathrm{H}\,|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$= \frac{1}{2}\big[|00\rangle \otimes (a|0\rangle + b|1\rangle) + |01\rangle \otimes (a|1\rangle + b|0\rangle)$$

$$+ |10\rangle \otimes (a|0\rangle - b|1\rangle) + |11\rangle \otimes (a|1\rangle - b|0\rangle)\big]$$

Quantum Teleportation — Step 5



$$|\Psi\rangle_4 = \frac{1}{2}\Big[\,|00\rangle\,(a|0\rangle + b|1\rangle) + |01\rangle\,(a|1\rangle + b|0\rangle) + |10\rangle\,(a|0\rangle - b|1\rangle) + |11\rangle\,(a|1\rangle - b|0\rangle)\Big]$$

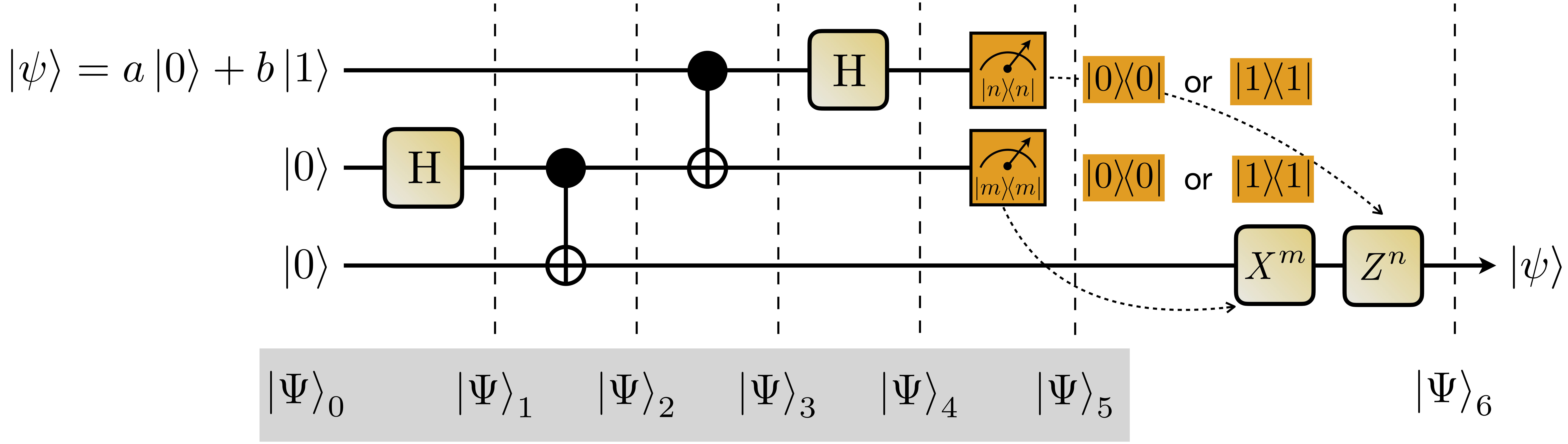
$$\Pi_{00} = |0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes I = |00\rangle\langle 00| \otimes I \qquad p_{00} = \langle\Psi|_4\,[\Pi_{00}\otimes I]\,|\Psi\rangle_4 = 1/4$$

$$\Pi_{01} = |0\rangle\langle 0| \otimes |1\rangle\langle 1| \otimes I = |01\rangle\langle 01| \otimes I \qquad p_{01} = \langle\Psi|_4\,[\Pi_{01}\otimes I]\,|\Psi\rangle_4 = 1/4$$

$$\Pi_{10} = |1\rangle\langle 1| \otimes |0\rangle\langle 0| \otimes I = |10\rangle\langle 10| \otimes I \qquad p_{10} = \langle\Psi|_4\,[\Pi_{10}\otimes I]\,|\Psi\rangle_4 = 1/4$$

$$\Pi_{11} = |1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes I = |11\rangle\langle 11| \otimes I \qquad p_{11} = \langle\Psi|_4\,[\Pi_{11}\otimes I]\,|\Psi\rangle_4 = 1/4$$

Quantum Teleportation — Step 5



$$|\Psi\rangle_4 = \frac{1}{2}\Big[\,|00\rangle\,(a\,|0\rangle + b\,|1\rangle) + |01\rangle\,(a\,|1\rangle + b\,|0\rangle) + |10\rangle\,(a\,|0\rangle - b\,|1\rangle) + |11\rangle\,(a\,|1\rangle - b\,|0\rangle)\Big]$$

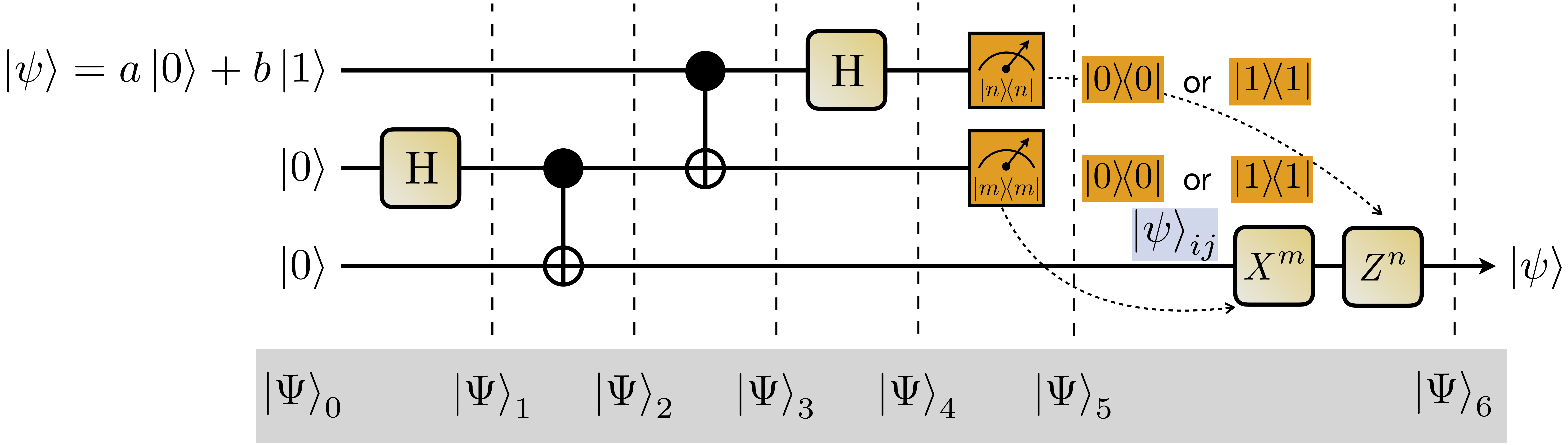
$$|\Psi\rangle_5^{(00)} = \frac{\Pi_{00}\,|\Psi\rangle_4}{\sqrt{p_{00}}} = |\cancel{00}\rangle \otimes (a|0\rangle + b|1\rangle)$$

$$|\Psi\rangle_5^{(10)} = \frac{\Pi_{10}\,|\Psi\rangle_4}{\sqrt{p_{10}}} = |\cancel{10}\rangle \otimes (a|0\rangle - b|1\rangle)$$

$$|\Psi\rangle_5^{(01)} = \frac{\Pi_{01}\,|\Psi\rangle_4}{\sqrt{p_{01}}} = |\cancel{01}\rangle \otimes (a|1\rangle + b|0\rangle)$$

$$|\Psi\rangle_5^{(11)} = \frac{\Pi_{11}\,|\Psi\rangle_4}{\sqrt{p_{11}}} = |\cancel{11}\rangle \otimes (a|1\rangle - b|0\rangle)$$

Quantum Teleportation — Step 6



nm

$$00 \longrightarrow X^0 Z^0 \,|\psi\rangle_{00} = \mathbb{1}\,|\psi\rangle_{00} = \mathbb{1}\,(a\,|0\rangle + b\,|1\rangle) = a\,|0\rangle + b\,|1\rangle = |\psi\rangle \quad ✓$$

$$01 \longrightarrow X^1 Z^0 \,|\psi\rangle_{01} = X\,|\psi\rangle_{01} = X(a\,|1\rangle + b\,|0\rangle) = a\,|0\rangle + b\,|1\rangle = |\psi\rangle \quad ✓$$

$$10 \longrightarrow X^0 Z^1 \,|\psi\rangle_{10} = Z\,|\psi\rangle_{10} = Z(a\,|0\rangle - b\,|1\rangle) = a\,|0\rangle + b\,|1\rangle = |\psi\rangle \quad ✓$$

$$11 \longrightarrow X^1 Z^1 \,|\psi\rangle_{11} = XZ\,|\psi\rangle_{11} = XZ(a\,|1\rangle - b\,|0\rangle) = a\,|0\rangle + b\,|1\rangle = |\psi\rangle \quad ✓$$

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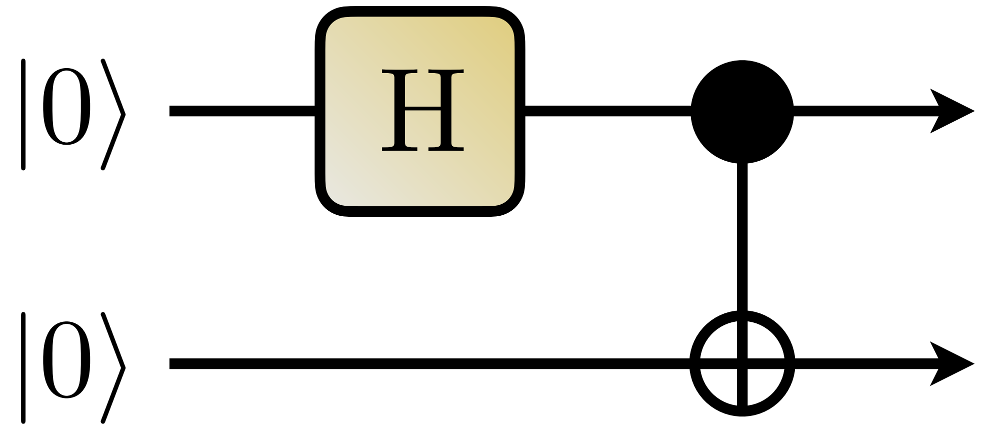
Resource State for Teleportation

Resource state for teleportation:



(Bell state)

$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

Composite states: $|\psi\rangle^{(1)} = \begin{bmatrix}c_0\\c_1\end{bmatrix} = c_0|0\rangle + c_1|1\rangle$ $\quad |\psi\rangle^{(2)} = \begin{bmatrix}d_0\\d_1\end{bmatrix} = d_0|0\rangle + d_1|1\rangle$

$$|\Psi\rangle = |\psi\rangle^{(1)} \otimes |\psi\rangle^{(2)} = \begin{bmatrix}c_0\\c_1\end{bmatrix} \otimes \begin{bmatrix}d_0\\d_1\end{bmatrix} = \begin{bmatrix}c_0\begin{bmatrix}d_0\\d_1\end{bmatrix}\\c_1\begin{bmatrix}d_0\\d_1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}c_0d_0\\c_0d_1\\c_1d_0\\c_1d_1\end{bmatrix}$$

Can we always decompose a given vector into a tensor product of vectors?

$$\begin{bmatrix}\star\\\star\end{bmatrix} \otimes \begin{bmatrix}\star\\\star\end{bmatrix} \overset{?}{\longleftarrow} \begin{bmatrix}\star\\\star\\\star\\\star\end{bmatrix}$$

Answer: **No**!

What Entanglement Is

A composite quantum state that **cannot** be written as a tensor product of two smaller quantum states is called **entangled** state

$$|\Psi\rangle \neq |\psi\rangle^{(1)} \otimes |\psi\rangle^{(2)}$$

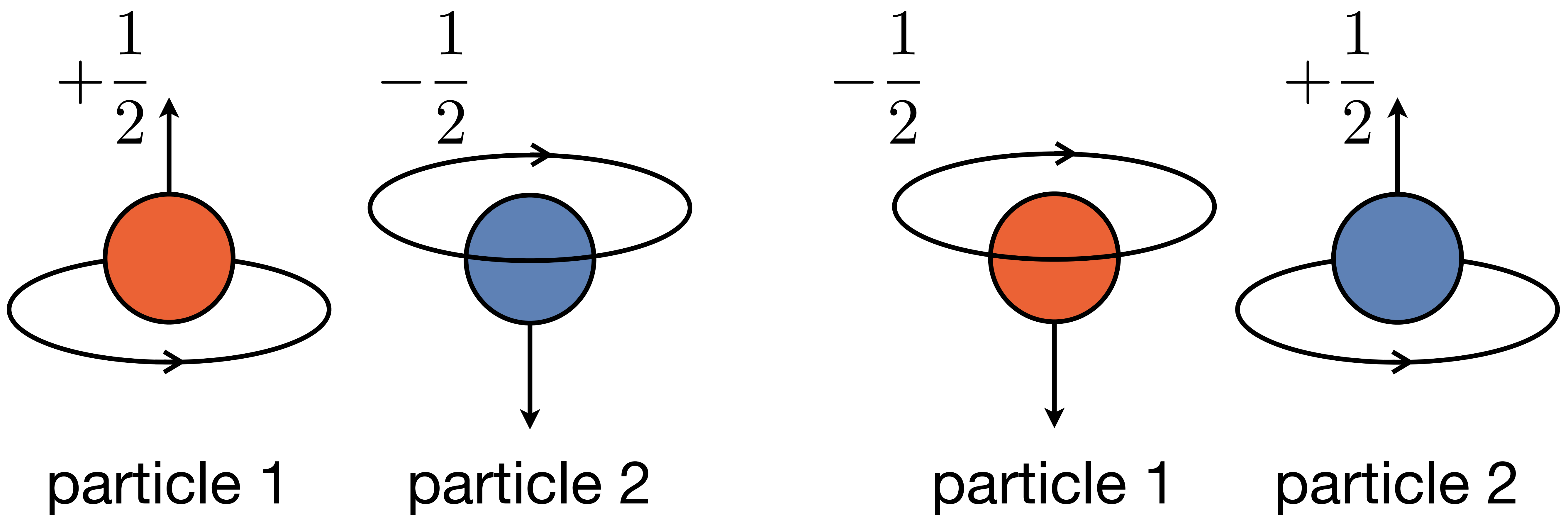
Otherwise it is called **separable** or **product** state.

- **Entangled systems share a common property**, but we don't know which part has which share until we measure it.
- For example, **two particles have a total (sum) spin of zero**, but we don't know the spin of each individual particle before we measure it.

$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$|0\rangle : +\frac{1}{2}$ spin

$|1\rangle : -\frac{1}{2}$ spin



What Entanglement is Not

Far Apart, 2 Particles Respond Faster Than Light

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Spooky! Quantum Action Is 10,000 Times Faster Than Light

Can quantum entanglement send info faster than light? Yes.
Einstein was wrong on this one. The link between two electrons vibrating in unison does send information faster than light. But Einstein still has the last laugh, because...

7:48 AM · Nov 17, 2018 · Twitter Web App

BREAKING—Successful faster-than-speed-of-light demonstration of **QUANTUM TELEPORTATION** of up to 44 kilometer. @Caltech & @Fermilab scientists teleported quantum info for a sustained period across distance of 44 km via quantum entanglement teleportation.
inqnet.caltech.e

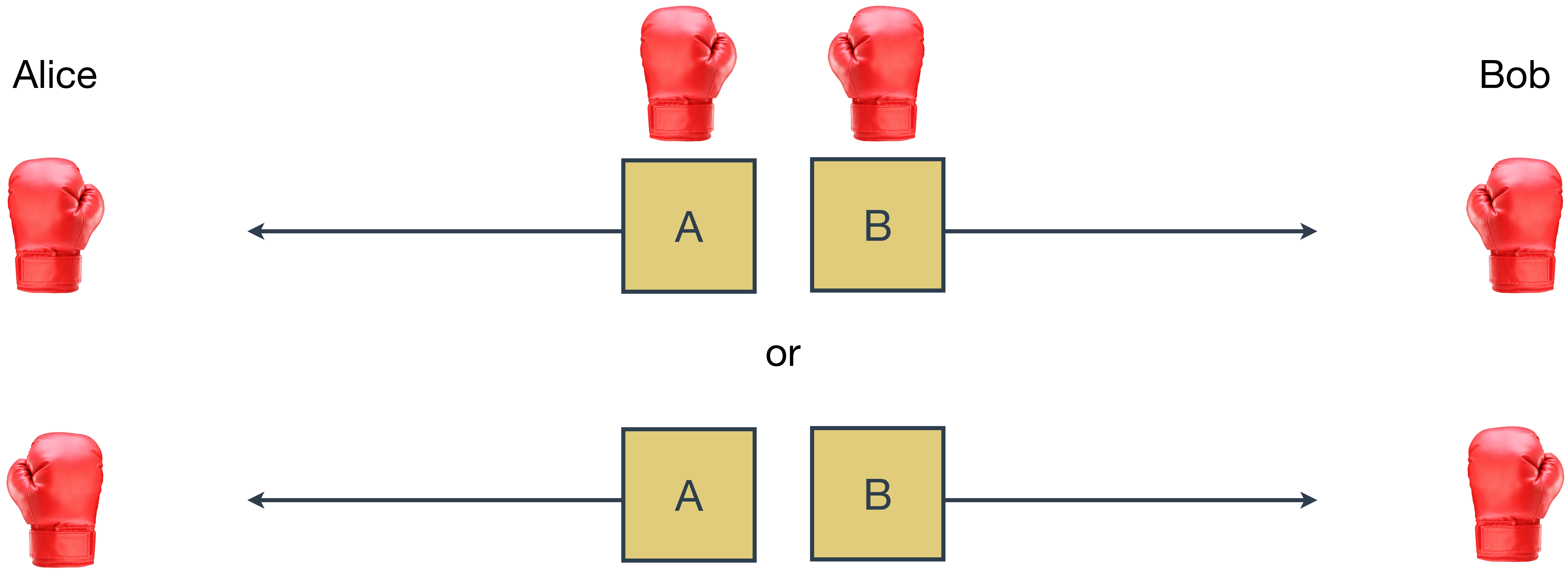
Quantum weirdness wins again: Entanglement clocks in at 10,000+ times faster than light

Quantum "spooky action at a distance" travels at least 10,000 times faster than light

NASA scientists achieve long-distance 'quantum teleportation' over 27 miles for the first time – paving the way for unhackable networks that transfer data faster than the speed of light

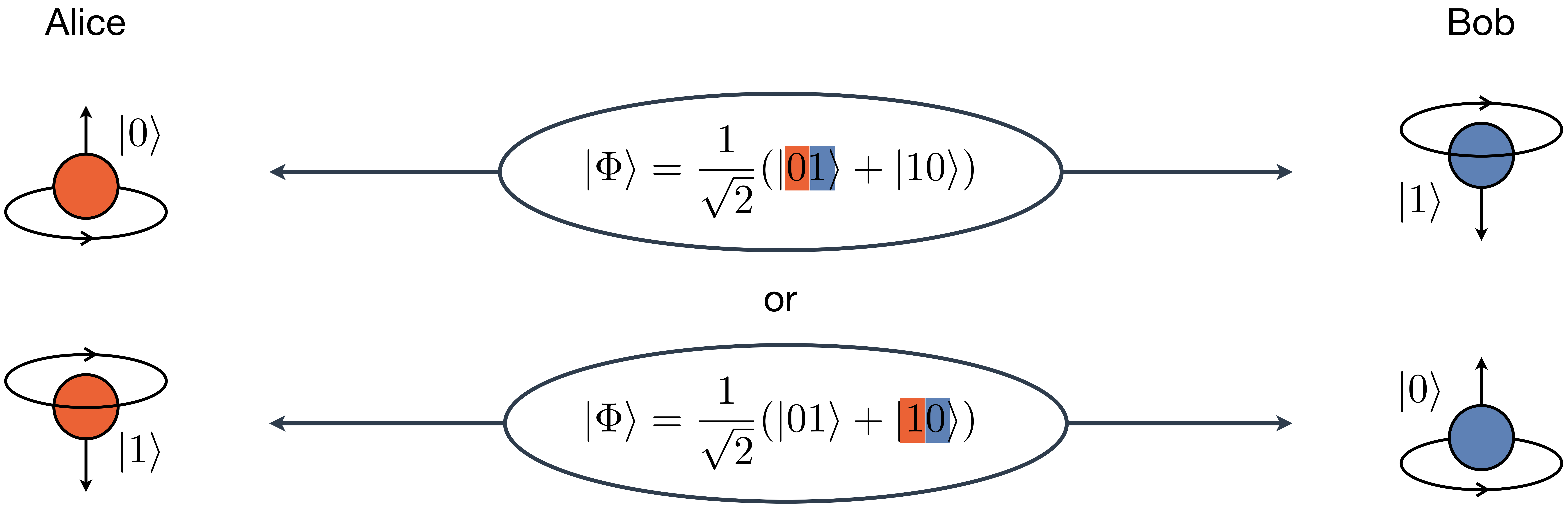
- Entanglement **DOES NOT** allow faster-than-light communication.
- Entanglement **DOES NOT** contain information. It contains correlations about information.

Classical Correlations



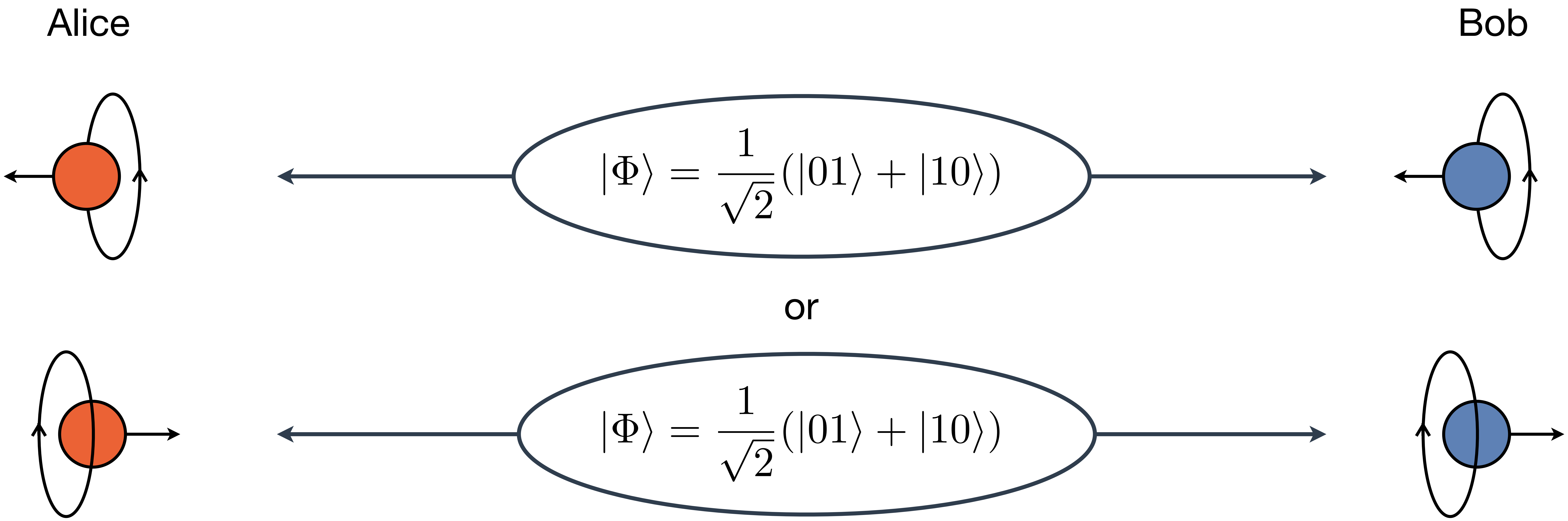
The gloves are **perfectly correlated** but **no information has traveled** from one place to another!

Quantum Correlations

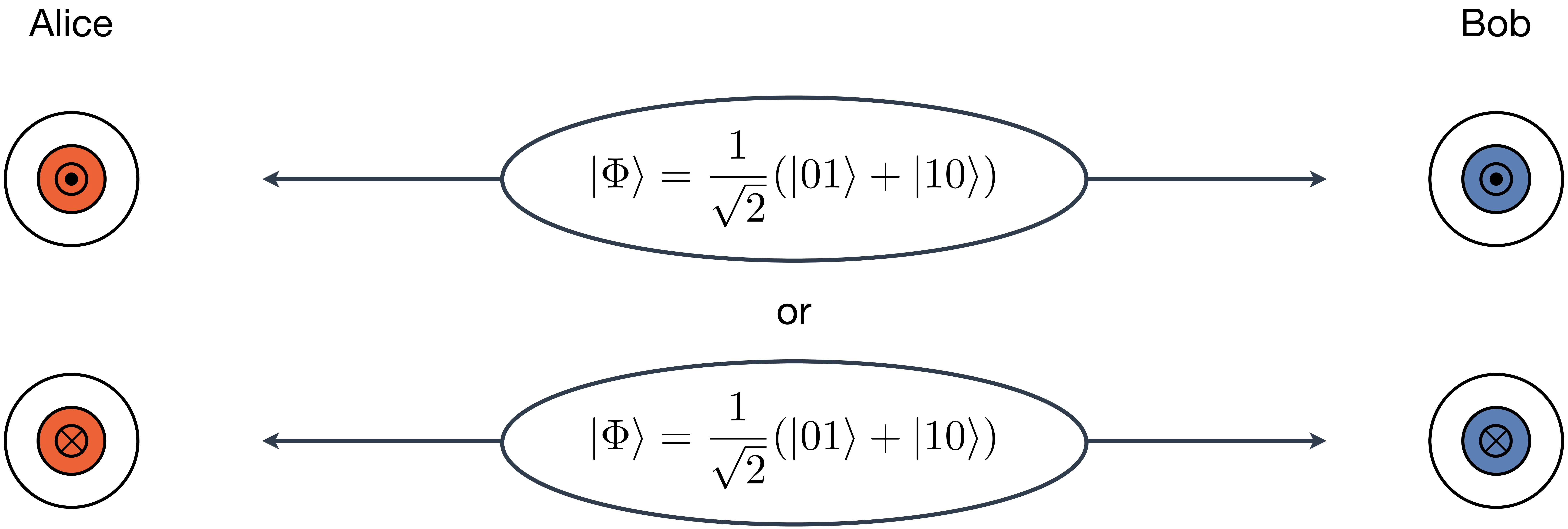


- The spins are **perfectly correlated** but **no information has traveled** from one place to another!
- So, what how are quantum correlations different from classical ones?

Quantum Correlations



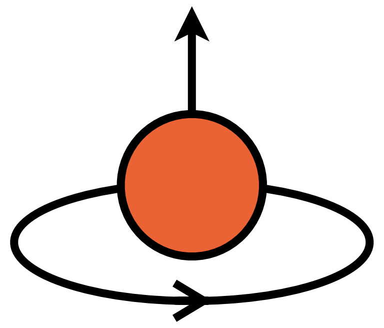
Quantum Correlations



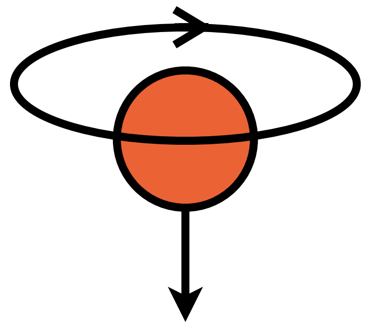
Quantum Correlations

$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

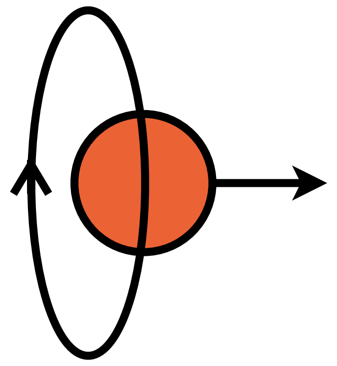
$$|0\rangle = |\uparrow\rangle = \begin{bmatrix}1\\0\end{bmatrix}$$



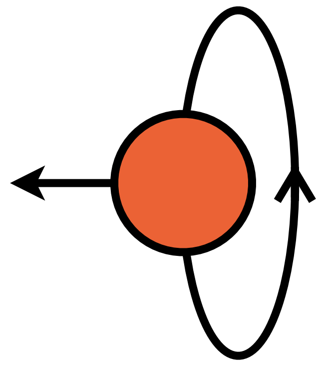
$$|1\rangle = |\downarrow\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$



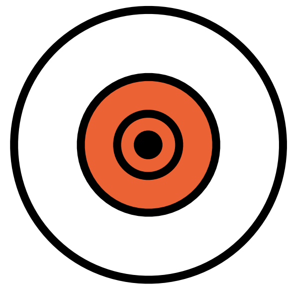
$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$$



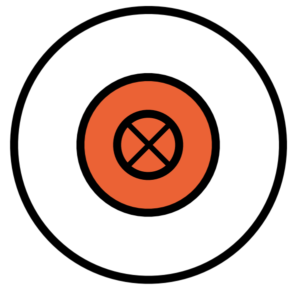
$$|\leftarrow\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$$



$$|\odot\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\\imath\end{bmatrix} = \frac{1}{\sqrt{2}}(|\uparrow\rangle + \imath\,|\downarrow\rangle)$$



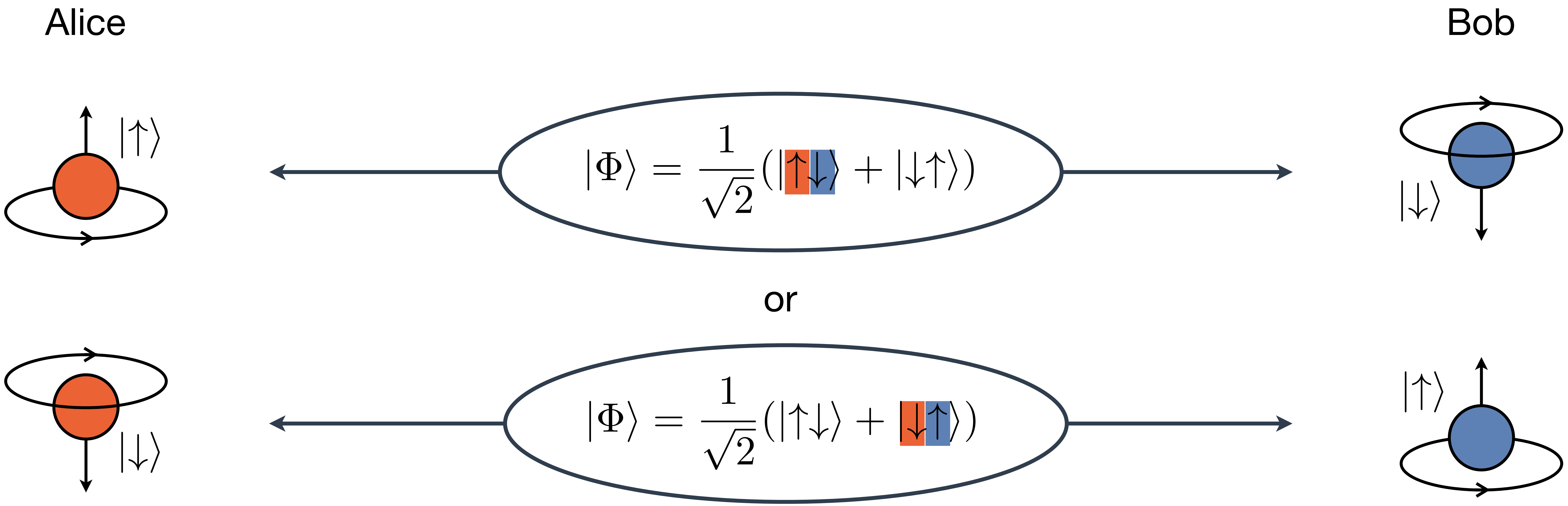
$$|\otimes\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-\imath\end{bmatrix} = \frac{1}{\sqrt{2}}(|\uparrow\rangle - \imath\,|\downarrow\rangle)$$



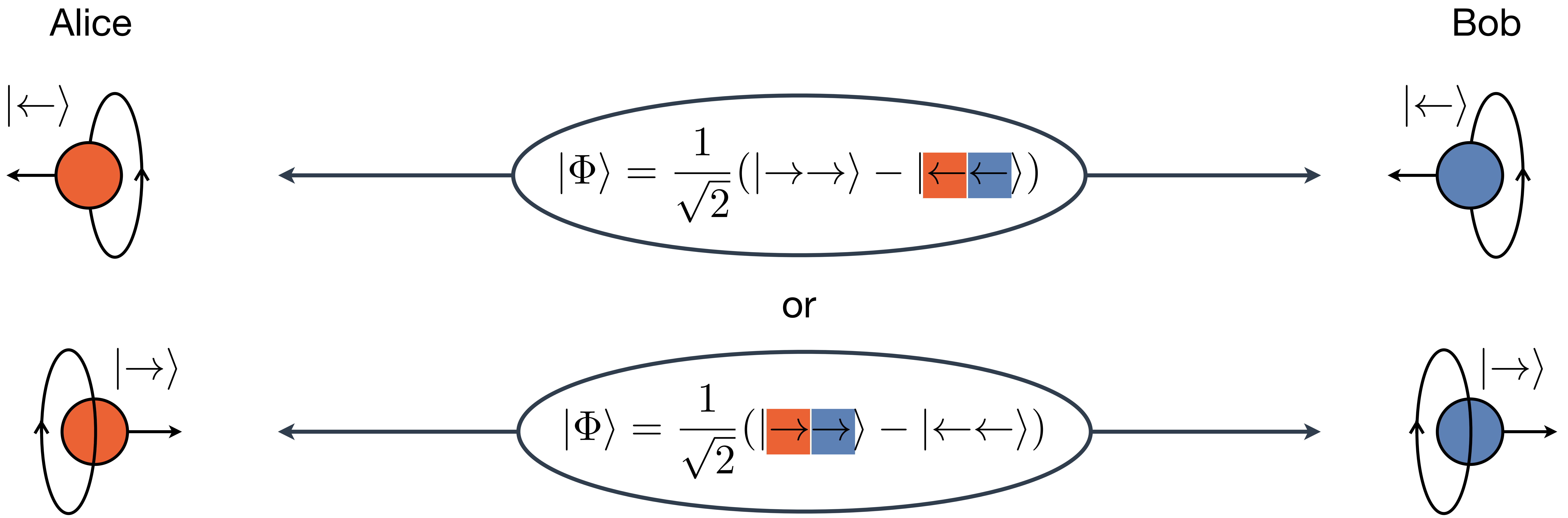
$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|\rightarrow\rightarrow\rangle - |\leftarrow\leftarrow\rangle)$$

$$|\Phi\rangle = \frac{-\imath}{\sqrt{2}}(|\odot\odot\rangle - |\otimes\otimes\rangle)$$

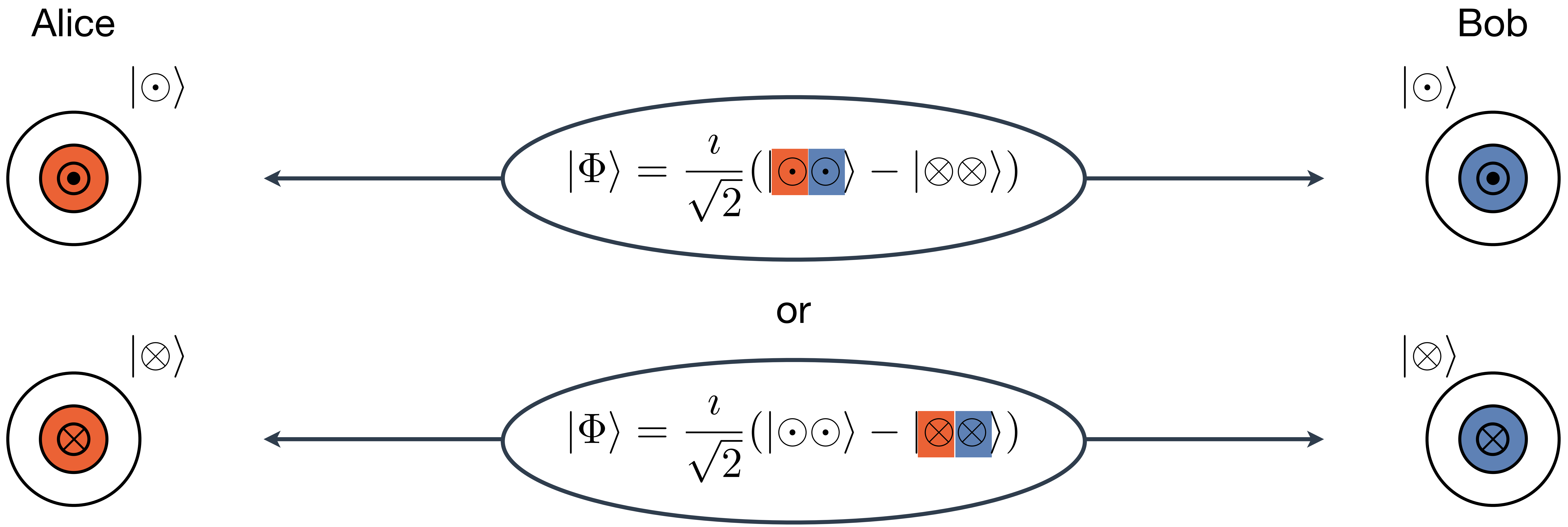
Quantum Correlations



Quantum Correlations

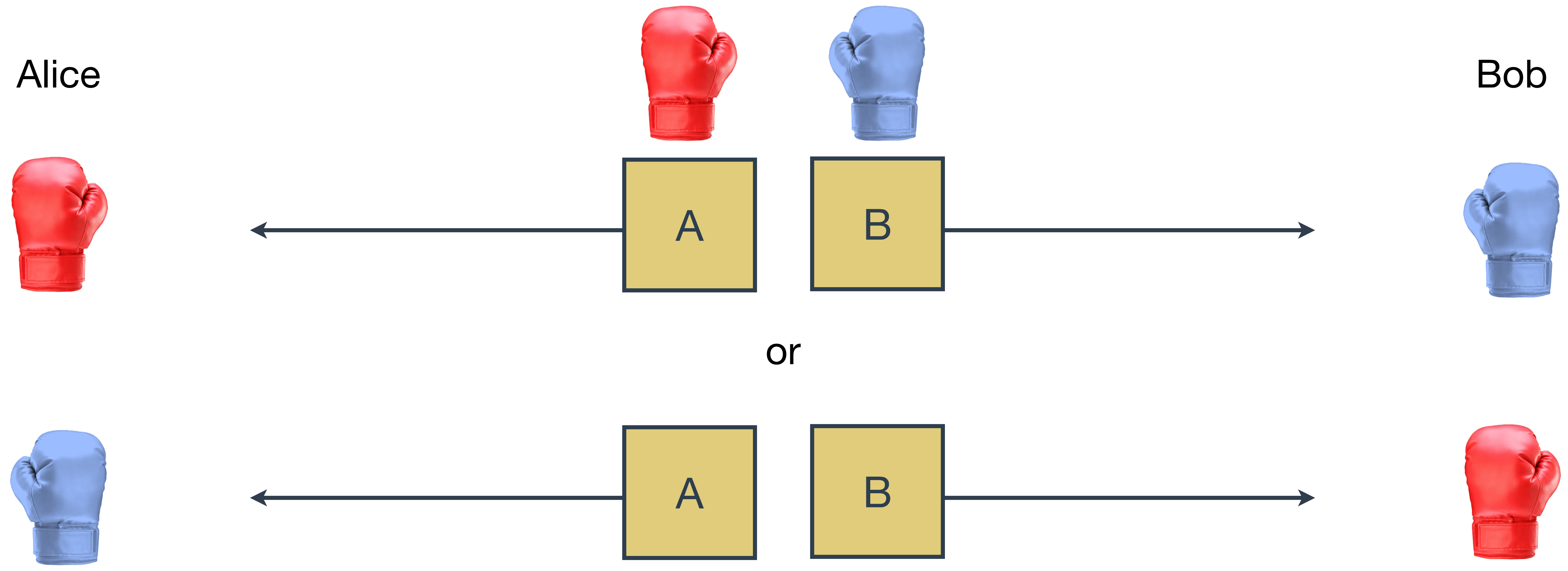


Quantum Correlations



- Quantum correlations are **stronger** than classical correlations
- In quantum systems **multiple properties** can be **simultaneously correlated**

Classical Correlations

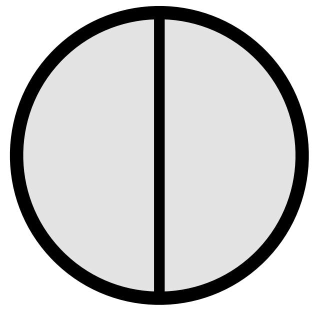


- **Classical properties are compatible**, i.e., they can be simultaneously measured
- **Quantum properties can be incompatible**, i.e., measuring one gives no information about the other

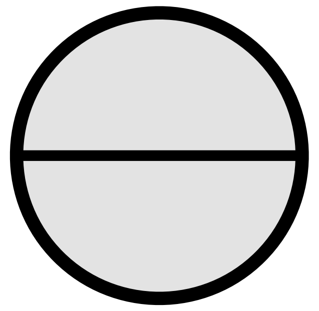
Quantum Correlations

$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}(|\updownarrow\leftrightarrow\rangle + |\updownarrow\leftrightarrow\rangle)$$

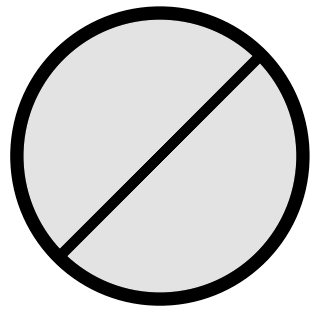
$$|0\rangle = |\updownarrow\rangle = \begin{bmatrix}1\\0\end{bmatrix}$$



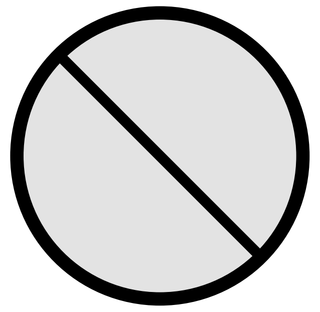
$$|1\rangle = |\leftrightarrow\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$



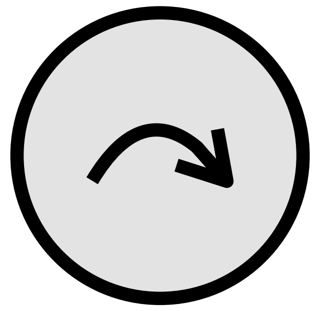
$$|\swarrow\!\!\!\!\nearrow\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}(|\updownarrow\rangle + |\leftrightarrow\rangle)$$



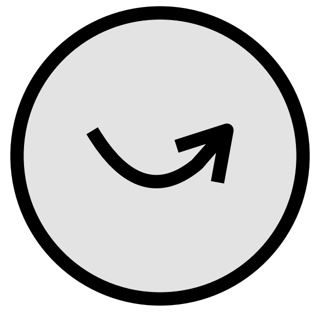
$$|\nwarrow\!\!\!\!\searrow\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{\sqrt{2}}(|\updownarrow\rangle - |\leftrightarrow\rangle)$$



$$|\circlearrowright\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\\imath\end{bmatrix} = \frac{1}{\sqrt{2}}(|\updownarrow\rangle + \imath\,|\leftrightarrow\rangle)$$



$$|\circlearrowleft\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-\imath\end{bmatrix} = \frac{1}{\sqrt{2}}(|\updownarrow\rangle - \imath\,|\leftrightarrow\rangle)$$



$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|\swarrow\!\!\!\!\nearrow\swarrow\!\!\!\!\nearrow\rangle - |\nwarrow\!\!\!\!\searrow\nwarrow\!\!\!\!\searrow\rangle)$$

$$|\Phi\rangle = \frac{-\imath}{\sqrt{2}}(|\circlearrowright\circlearrowright\rangle - |\circlearrowleft\circlearrowleft\rangle)$$

Characterizing Entanglement

Schmidt decomposition: $|\Psi\rangle = \sum_i \sqrt{\lambda_i} |e_i\rangle \otimes |h_i\rangle$

$\{|e_i\rangle, |h_i\rangle\}$: orthonormal basis

$\{\lambda_0, \lambda_1\}$: Schmidt coefficients

- The number of non-zero Schmidt coefficients **identifies entanglement**

$\lambda_0 = 1$ & $\lambda_1 = 0 \Rightarrow |\Psi\rangle = |e_0\rangle \otimes |h_0\rangle$ (separable)

$\lambda_0 \neq 0$ & $\lambda_1 \neq 0 \Rightarrow |\Psi\rangle = \lambda_0 |e_0\rangle \otimes |h_0\rangle + \lambda_1 |e_1\rangle \otimes |h_1\rangle$ (entangled)

- Entanglement is the **superposition** of composite quantum systems

$|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ $\quad \lambda_0 = \lambda_1 = \frac{1}{\sqrt{2}}$ $\quad |e_0^{(1)}\rangle = |e_0^{(2)}\rangle = |0\rangle$ $\quad |e_1^{(1)}\rangle = |e_1^{(2)}\rangle = |1\rangle$

$|\Psi\rangle = |00\rangle$ $\quad \lambda_0 = 1$ & $\lambda_1 = 0$ $\quad |e_0^{(1)}\rangle = |e_0^{(2)}\rangle = |0\rangle$ $\quad |e_1^{(1)}\rangle = |e_1^{(2)}\rangle = |1\rangle$

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- Entanglement is the **superposition** of composite quantum systems

$$|\Psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle) \qquad \lambda_0 = \lambda_1 = \frac{1}{\sqrt{2}}$$

$$|e_0^{(1)}\rangle = |0\rangle \qquad |e_0^{(2)}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|e_1^{(1)}\rangle = |1\rangle \qquad |e_1^{(2)}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Characterizing Entanglement

Schmidt decomposition: $|\Psi\rangle = \sum_i \sqrt{\lambda_i} |e_i\rangle \otimes |h_i\rangle$

$\{|e_i\rangle, |h_i\rangle\}$: orthonormal basis

$\{\lambda_0, \lambda_1\}$: Schmidt coefficients

- The number of non-zero Schmidt coefficients **identifies entanglement**

$\lambda_0 = 1$ & $\lambda_1 = 0 \Rightarrow |\Psi\rangle = |e_0\rangle \otimes |h_0\rangle$ (separable)

$\lambda_0 \neq 0$ & $\lambda_1 \neq 0 \Rightarrow |\Psi\rangle = \lambda_0 |e_0\rangle \otimes |h_0\rangle + \lambda_1 |e_1\rangle \otimes |h_1\rangle$ (entangled)

- Entanglement is the **superposition** of composite quantum systems

$|\Psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

$\lambda_0 = 1$ & $\lambda_1 = 0$

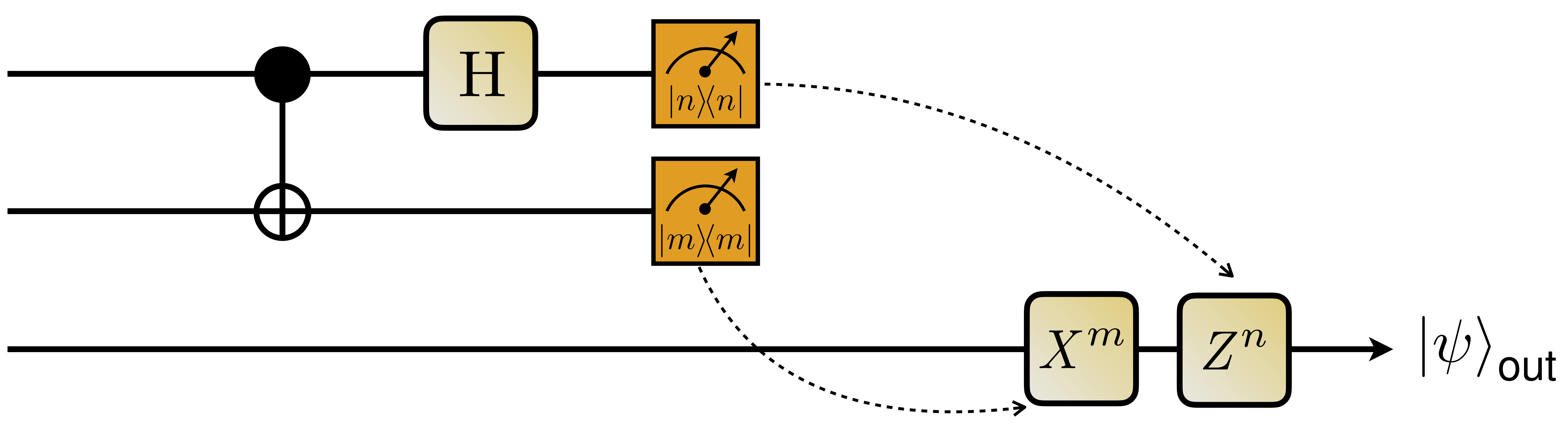
$|e_0^{(1)}\rangle = |e_0^{(2)}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$|e_1^{(1)}\rangle = |e_1^{(2)}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Teleportation for Different Resource States

$$|\psi\rangle_{\text{in}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|\Psi\rangle = \sqrt{1-x}\,|00\rangle + \sqrt{x}\,|11\rangle$$

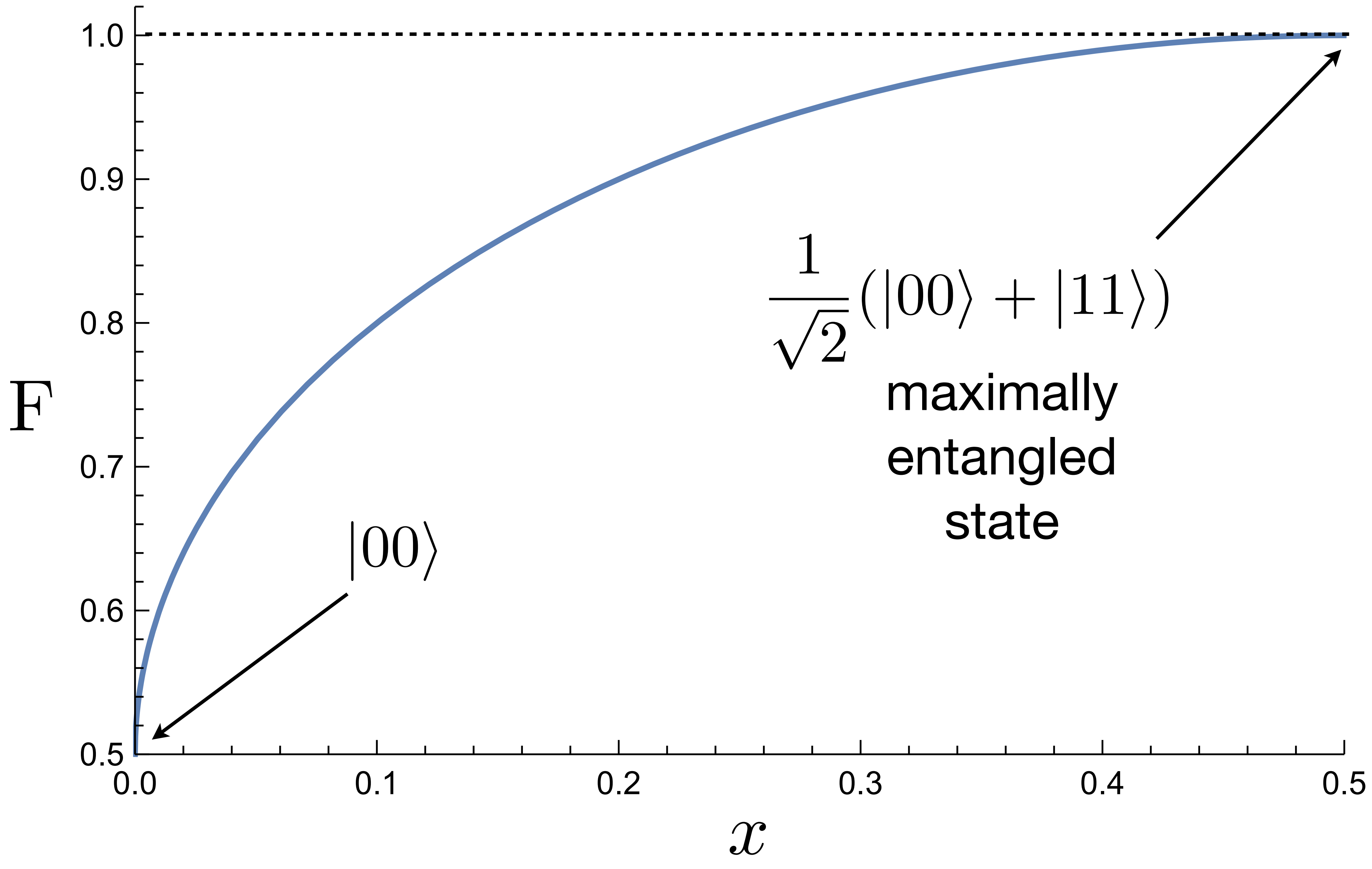


$x = 0 \Rightarrow |\Psi\rangle = |00\rangle$ (separable)

$x = \frac{1}{2} \Rightarrow |\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ (entangled)

Fidelity: $\mathrm{F} = |\langle\psi_{\text{in}}|\psi_{\text{out}}\rangle|^2 \quad 0 \leqslant \mathrm{F} \leqslant 1$

$$\mathrm{F} = 1 \Leftrightarrow |\psi_{\text{in}}\rangle = |\psi_{\text{out}}\rangle$$



Next Week

- Lecture 1 — Introduction to Quantum Systems (April 13, 2022)
- Lecture 2 — Teleportation and Entanglement (April 20, 2022)
- **Lecture 3 — Decoherence and Quantum Networks** (April 27, 2022)

slides can be found at: spyrostserkis.com

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Suggested Bibliography

- Quantum Mechanics
  - J. Townsend - “A Modern Approach to Quantum Mechanics”
  - L. Ballentine - “Quantum Mechanics”
- Quantum Information
  - J. Audretsch - “Entangled Systems”
  - M. Nielsen, I. Chuang - “Quantum Computation and Quantum Information”

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