Quantum Systems, Information, and Entanglement

Lecture 3. Decoherence and Quantum Networks

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HARVARD UNIVERSITY

NARANGLAB
MATERIALS THEORY AT HARVARD

Course Outline

- Lecture 1 — Introduction to Quantum Systems (April 13, 2022)
- Lecture 2 — Teleportation and Entanglement (April 20, 2022)
- **Lecture 3 — Decoherence and Quantum Networks** (April 27, 2022)

Lecture 3 — Decoherence and Quantum Networks

- Entanglement Swapping
  - Quantum Networks
- Decoherence
  - Pure VS Mixed States
  - Quantum Tomography
  - Quantification of Entanglement
- Quantum Networks With Repeaters
  - Entanglement Distillation
  - Quantum Key Distribution

Lecture 3 — Decoherence and Quantum Networks

- **Entanglement Swapping**
  - Quantum Networks
- Decoherence
  - Pure VS Mixed States
  - Quantum Tomography
  - Quantification of Entanglement
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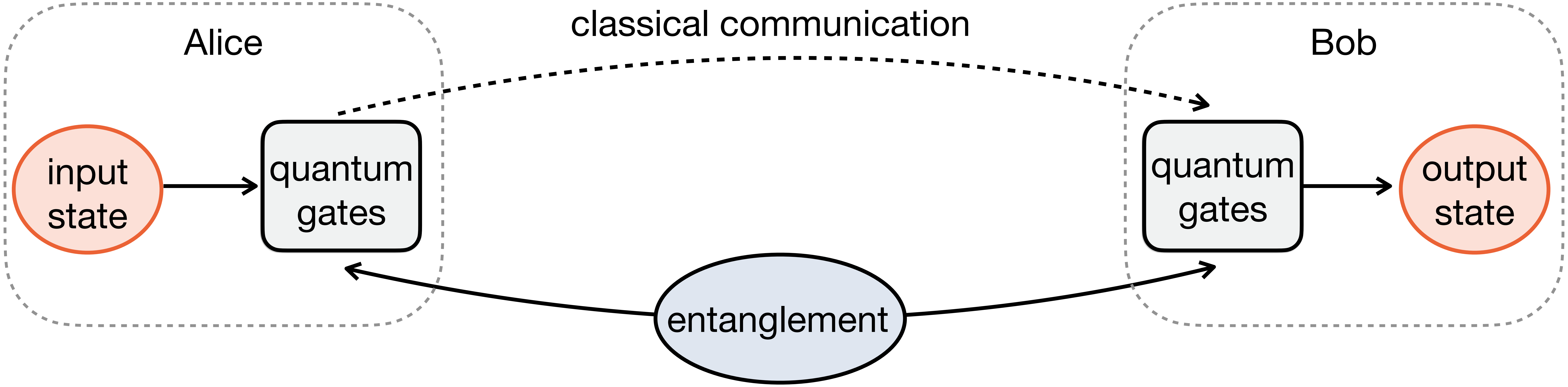
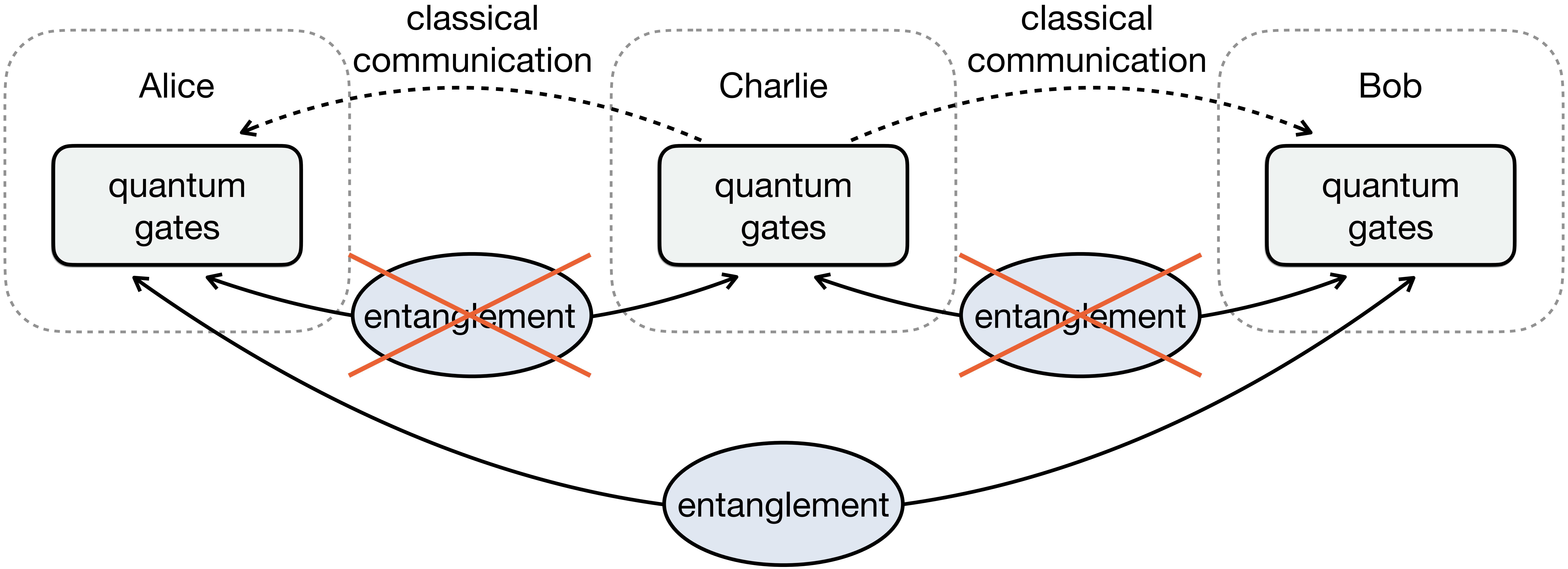
  - Quantum Key Distribution

Quantum Teleportation



C. H. Bennett, et al., Phys. Rev. Lett. 70, 1895 (1993)

Entanglement Swapping

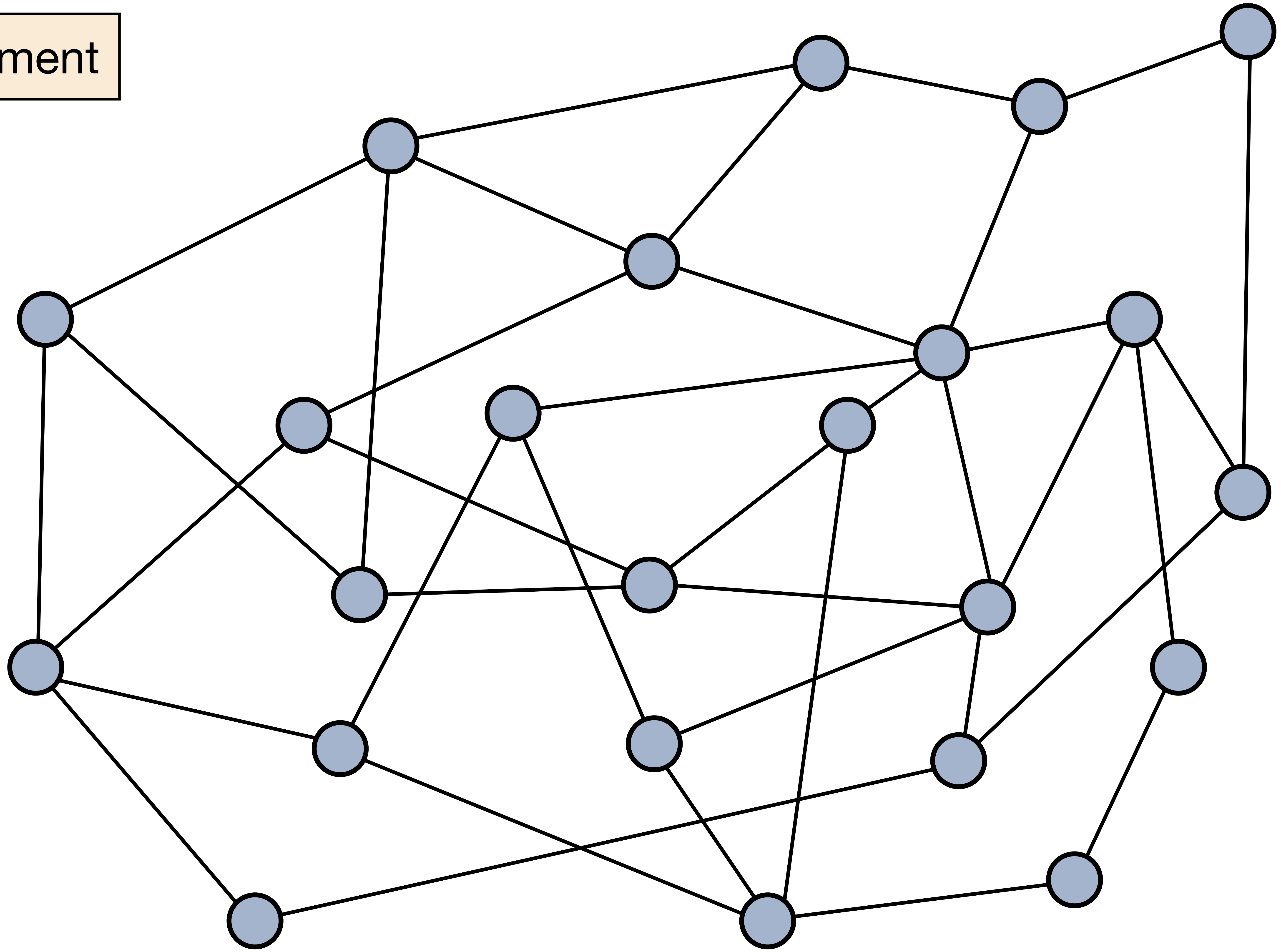


Entanglement swapping is effectively quantum **teleportation of an entangled state**

M. Żukowski et al., Phys. Rev. Lett. (1993)

Quantum Network

Task: Distribute entanglement

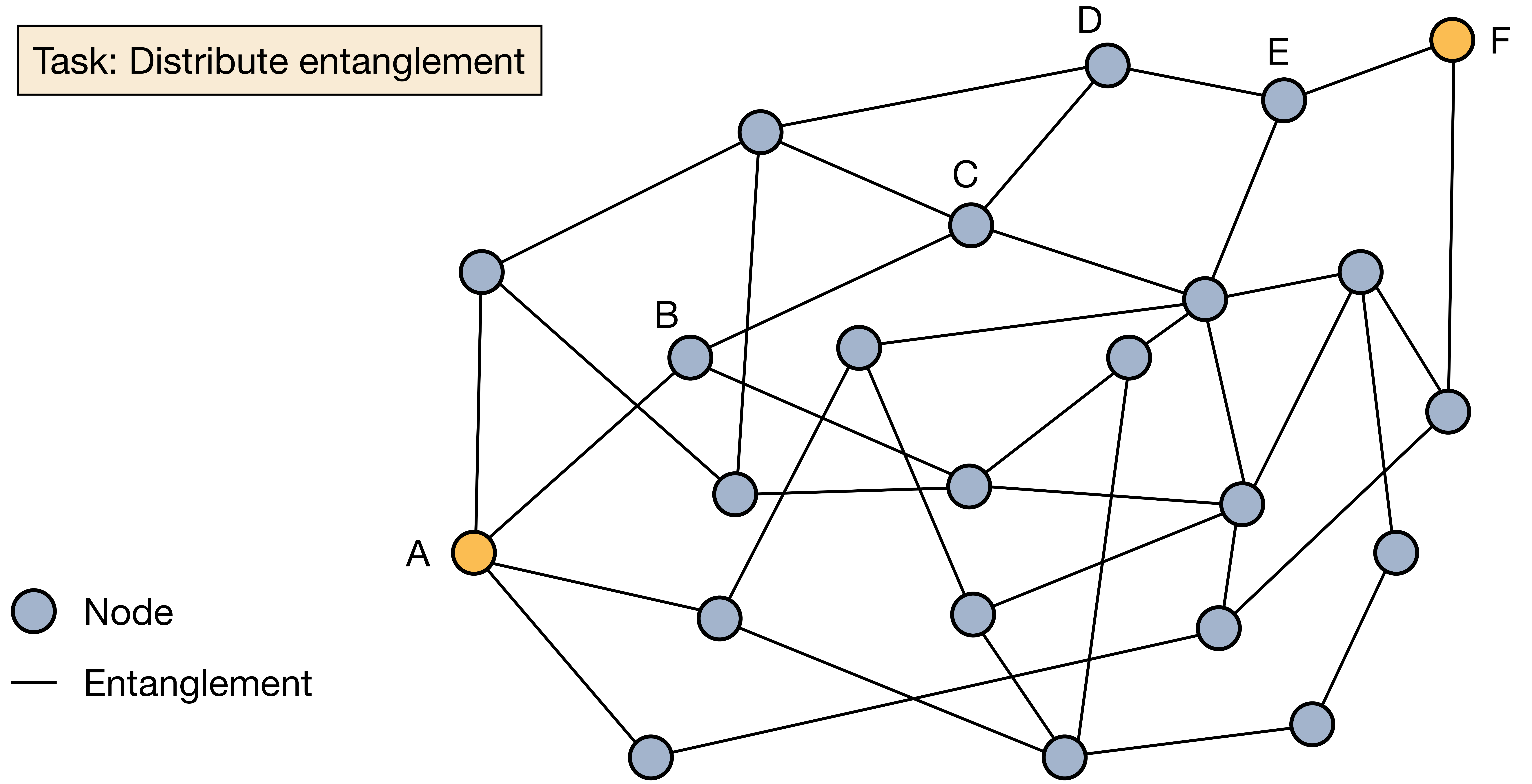


Node

Entanglement

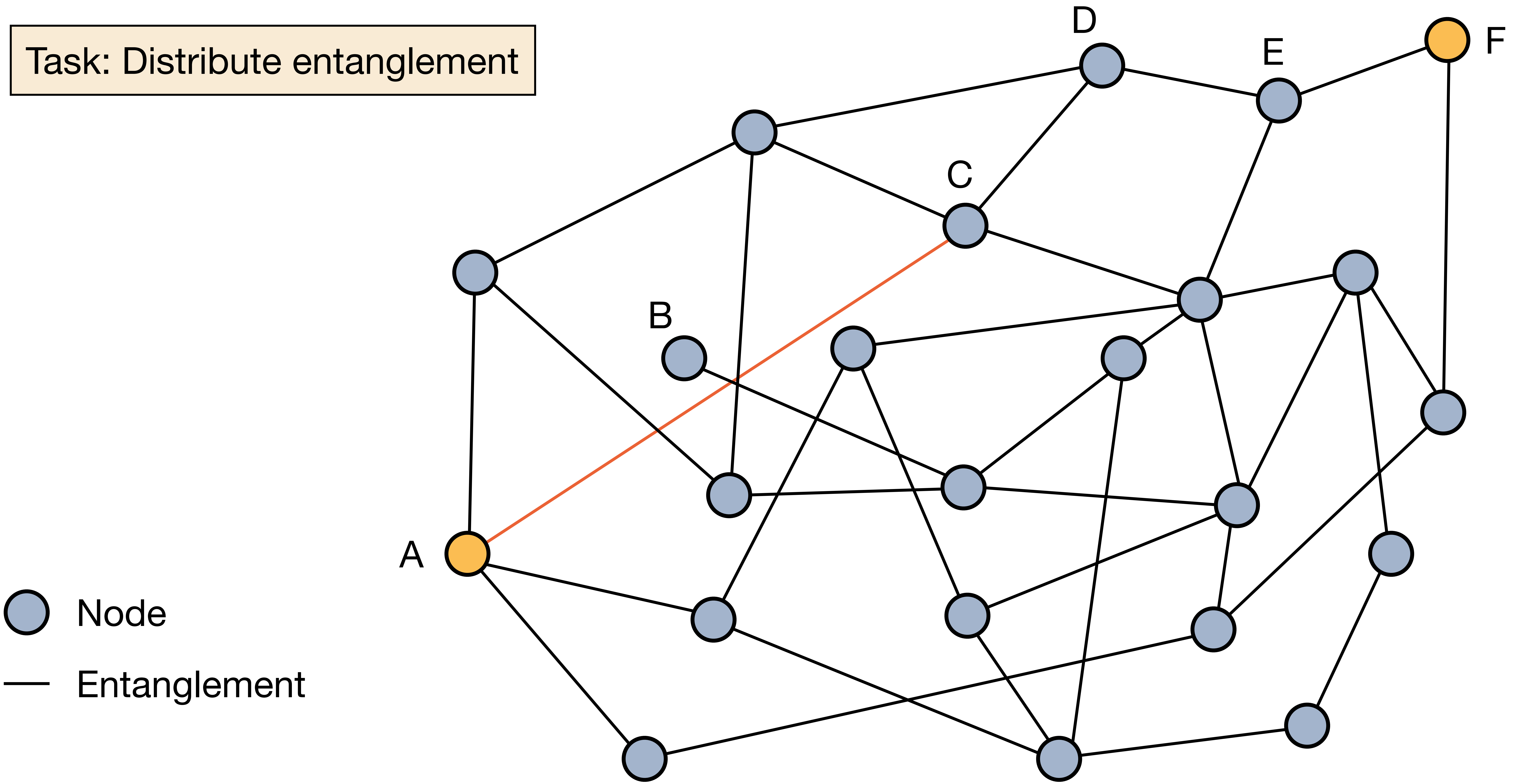
Quantum Network

Task: Distribute entanglement



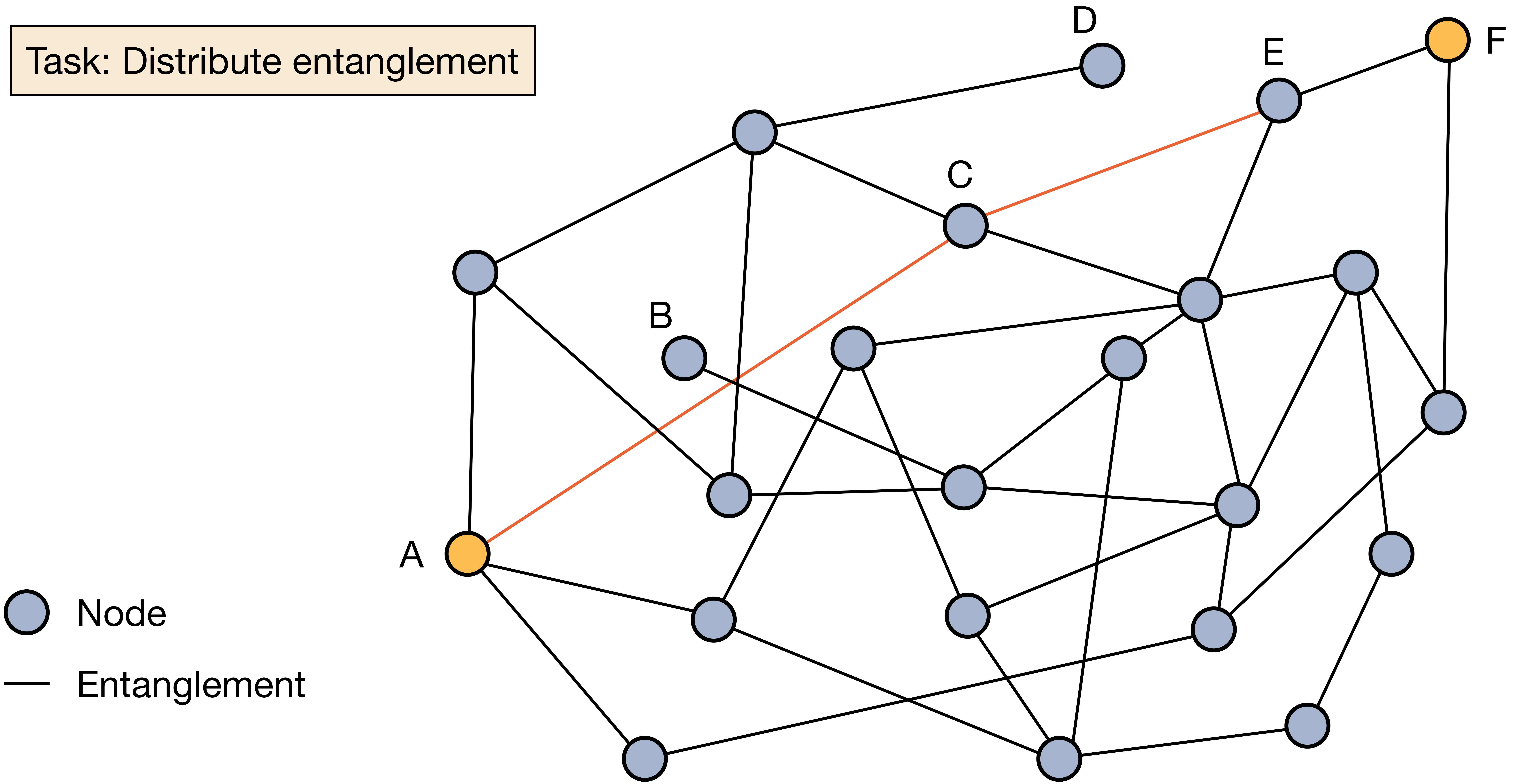
Quantum Network

Task: Distribute entanglement



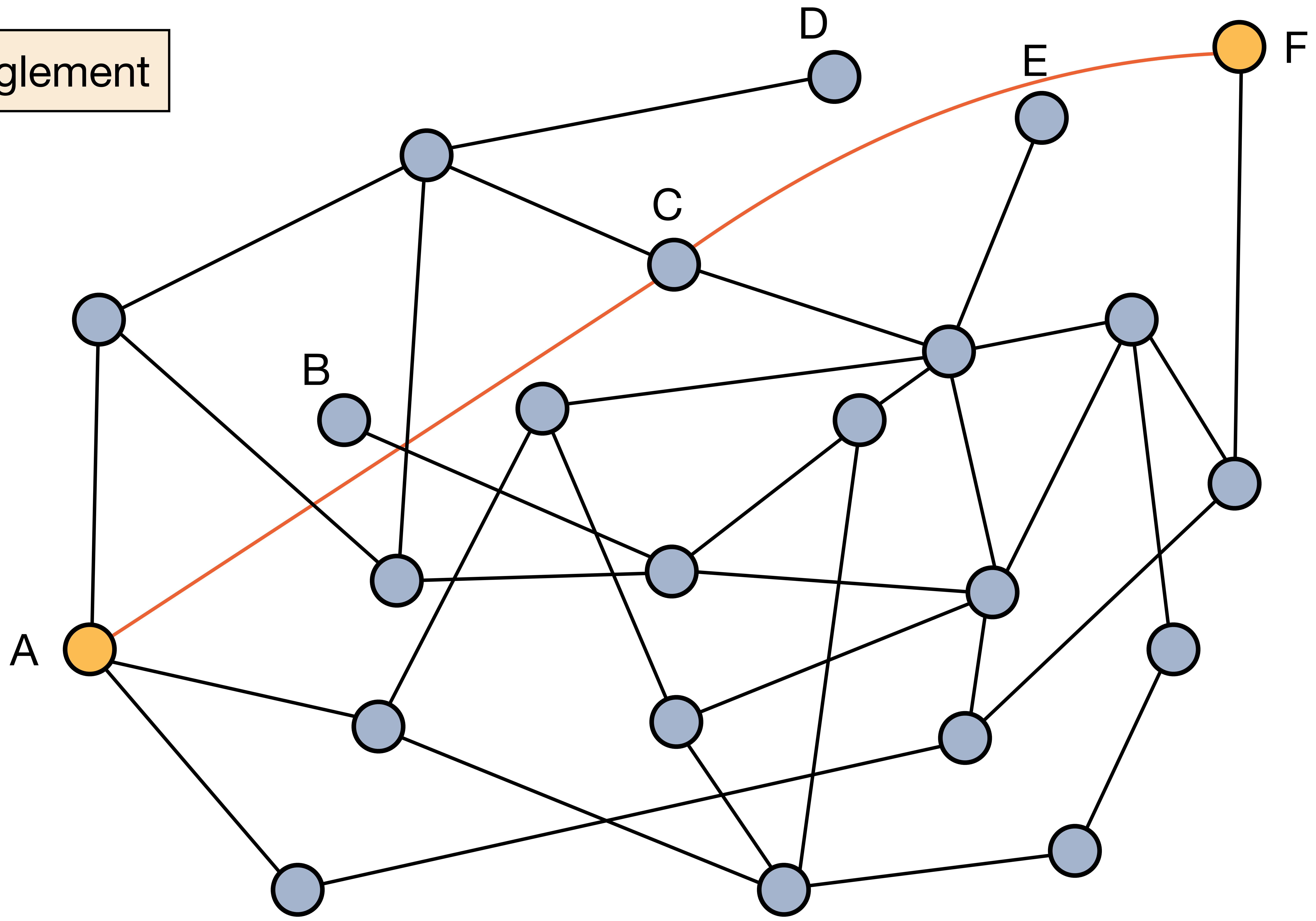
Quantum Network

Task: Distribute entanglement



Quantum Network

Task: Distribute entanglement

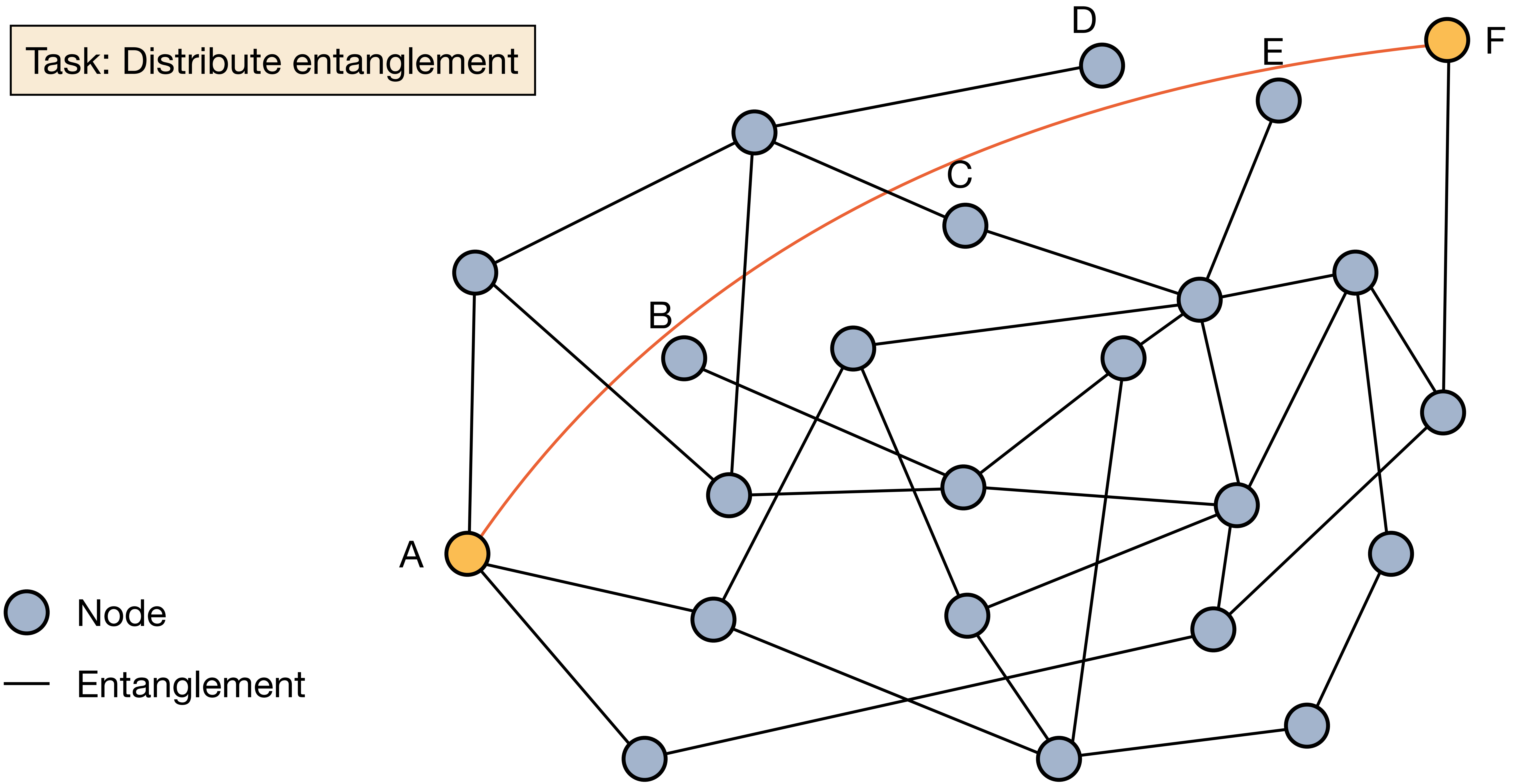


Node

Entanglement

Quantum Network

Task: Distribute entanglement



Lecture 3 — Decoherence and Quantum Networks

- Entanglement Swapping
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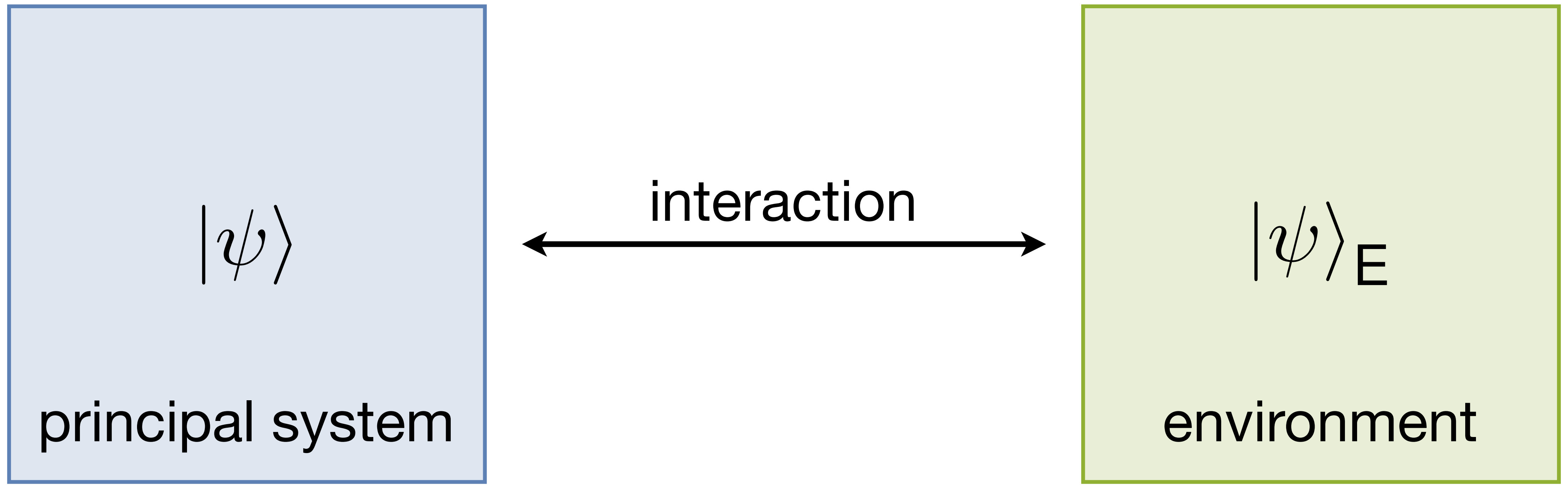
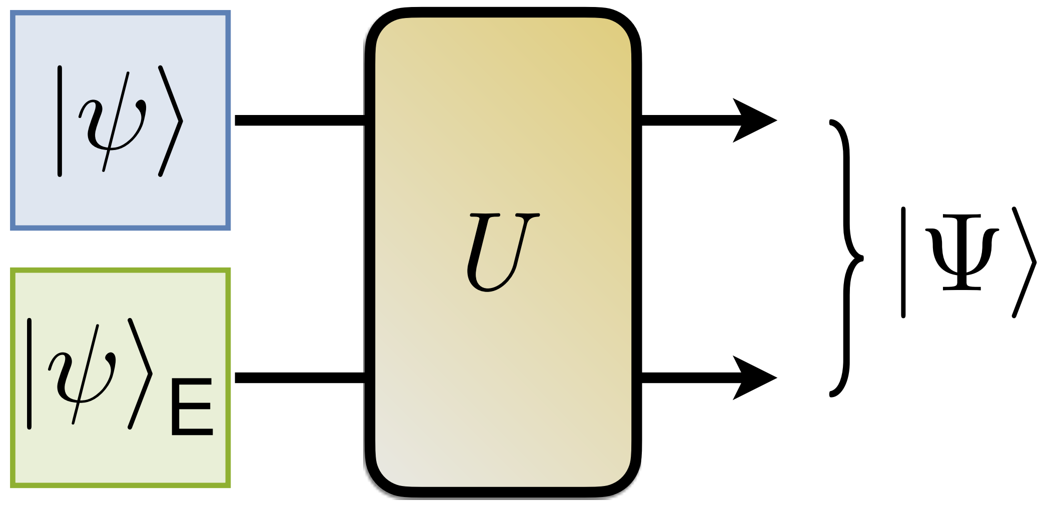
Noise in Quantum Systems

- Two quantum systems can **interact** with each other



- **Noise** induced into a quantum system can be modeled through the **Stinesping dilation**



$|\Psi\rangle$ is in general an entangled state $|\Psi\rangle \neq |\psi'\rangle \otimes |\psi'\rangle_E$

- Interaction in general **creates entangled quantum systems**

W. F. Stinespring, Proc. Am. Math. Soc. (1955)

Quantum States as Matrices

- We can represent quantum states with **matrices** $|\psi\rangle \rightarrow S = |\psi\rangle\langle\psi|$
  - For example $|\psi\rangle = a\,|0\rangle + b\,|1\rangle \rightarrow S = |\psi\rangle\langle\psi| = (a\,|0\rangle + b\,|1\rangle)(a^*\,\langle 0| + b^*\,\langle 1|) = \begin{bmatrix} |a| & ab^* \\ a^*b & |b| \end{bmatrix}$
- A **composite system of two quantum states** S_1 and S_2 can be written as $S_1 \otimes S_2$
  - For example $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ $\quad Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$

$$X \otimes Y = \begin{bmatrix} x_{11}Y & x_{12}Y \\ x_{21}Y & x_{22}Y \end{bmatrix} = \begin{bmatrix} x_{11}\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} & x_{12}\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \\ x_{21}\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} & x_{22}\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} x_{11}y_{11} & x_{11}y_{12} & x_{12}y_{11} & x_{12}y_{12} \\ x_{11}y_{21} & x_{11}y_{22} & x_{12}y_{21} & x_{12}y_{22} \\ x_{21}y_{11} & x_{21}y_{12} & x_{22}y_{11} & x_{22}y_{12} \\ x_{21}y_{21} & x_{21}y_{22} & x_{22}y_{21} & x_{22}y_{22} \end{bmatrix}$$

$$\begin{bmatrix} \star & \star \\ \star & \star \end{bmatrix} \otimes \begin{bmatrix} \star & \star \\ \star & \star \end{bmatrix} \xleftarrow{\;?\;} = \begin{bmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{bmatrix}$$

Not always

Reduced Quantum State

- The **trace** of a matrix $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ is $\quad \mathrm{tr}(X) = x_{11} + x_{22}$

- The **partial trace** of a product matrix $X \otimes Y$ is

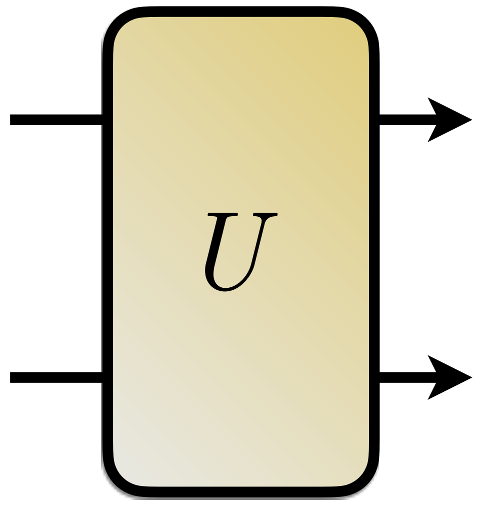
$$\mathrm{tr}_1(X \otimes Y) = Y \qquad\qquad \mathrm{tr}_2(X \otimes Y) = X$$

- The **partial trace** of an arbitrary matrix $M = \left[\begin{array}{c:c} A_{2\times 2} & B_{2\times 2} \\ \hdashline C_{2\times 2} & D_{2\times 2} \end{array}\right]$ is

$$\mathrm{tr}_1(M) = A + D \qquad\qquad \mathrm{tr}_2(M) = \begin{bmatrix} \mathrm{tr}(A) & \mathrm{tr}(B) \\ \mathrm{tr}(C) & \mathrm{tr}(D) \end{bmatrix}$$

- Given a composite quantum state $S^{(AB)}$ we can retrieve the **reduced quantum states** as follows

$$S^{(AB)} = S^{(A)} \otimes S^{(B)} \left\{ \begin{array}{l} S^{(A)} \\ \\ S^{(B)} \end{array} \right.$$



$$\left. \right\} S'^{(AB)}$$

$$S'^{(A)} = \mathrm{tr}_{\mathrm{B}}\left[S'^{(AB)}\right]$$ principal quantum system

$$S'^{(B)} = \mathrm{tr}_{\mathrm{A}}\left[S'^{(AB)}\right]$$ environment quantum system

Quantum Systems (Revisited)

A **quantum system**, known as **quantum state**, is represented by a ~~vector~~ **matrix**

- A quantum state is represented by a **normalized positive matrix** called the **density matrix** (positive → positive eigenvalues)
  - **Normalization** for density matrices implies $\mathrm{tr}(S) = 1$

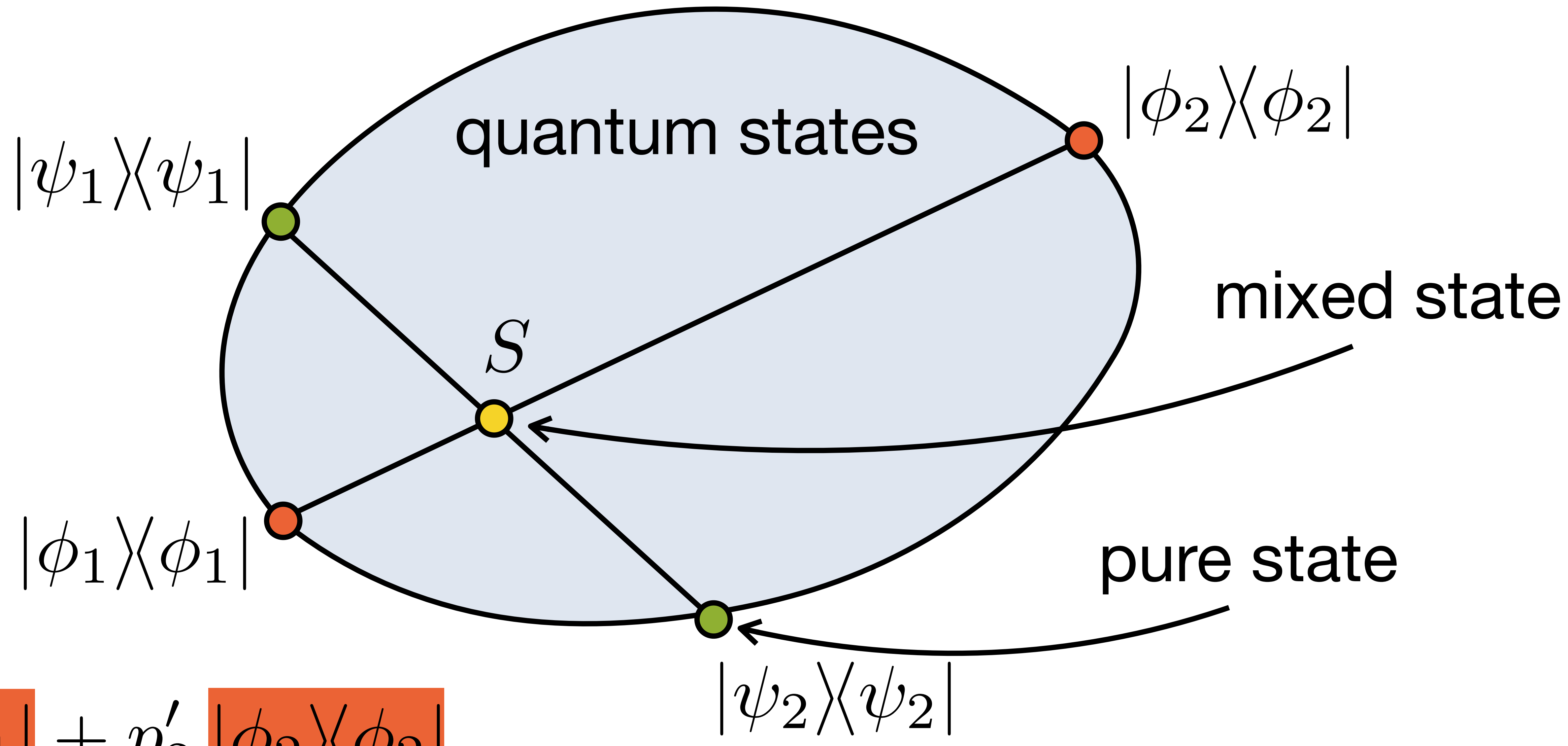
- A density matrix that is an outer product of a vector is called **pure quantum state** otherwise it is called **mixed quantum state**
  - Mixed quantum states can be written as

$$S = \sum_i p_i \left|\psi_i\right\rangle\!\left\langle\psi_i\right|$$

  - Evolution of a mixed state is given by

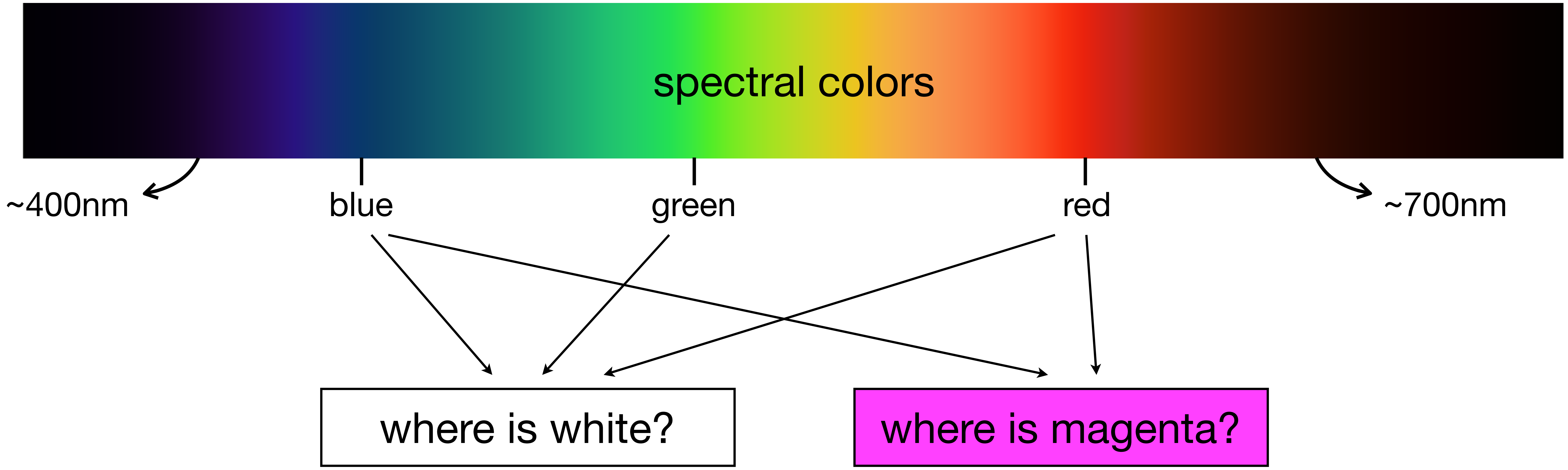
$$\left|\psi\right\rangle \rightarrow U\left|\psi\right\rangle \qquad \left|\psi\right\rangle\!\left\langle\psi\right| \rightarrow U\left|\psi\right\rangle\!\left\langle\psi\right| U^\dagger$$

$$\boxed{S \rightarrow USU^\dagger}$$



$$S = p_1 \left|\psi_1\right\rangle\!\left\langle\psi_1\right| + p_2 \left|\psi_2\right\rangle\!\left\langle\psi_2\right| = p_1' \left|\phi_1\right\rangle\!\left\langle\phi_1\right| + p_2' \left|\phi_2\right\rangle\!\left\langle\phi_2\right|$$

Color - State Analogy



- **Spectral colors** (e.g., red, green, and blue) correspond to specific wavelengths of light
  - Analogous to **pure** quantum states

- **Non-spectral colors** (e.g., white and magenta) correspond to combinations of spectral colors
  - Analogous to **mixed** quantum states

Reconstruction of a Quantum State

- It is **impossible to determine the unknown quantum state of a single individual system** by any measurement or sequence of measurements. We need to perform statistics on an ensemble of quantum systems, i.e., measuring different observables.

- The **mean value** of a **random variable** X is given by $\mathbb{E}(\mathrm{X}) = \sum_i x_i p(x_i)$

probability of the event

value of the event

- The **mean value** of an **observable** X is given by $\mathbb{E}(X) = \sum_i x_i p(x_i)$

probability of the measurement

outcome of the measurement

$$S = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| = \frac{1}{d}\mathbb{1} + \frac{1}{2}\sum_{i=1}^{d^2-1} \mathbb{E}(M_i) M_i$$

For qubits:

$$M_1 = |0\rangle\langle 1| + |1\rangle\langle 0|$$

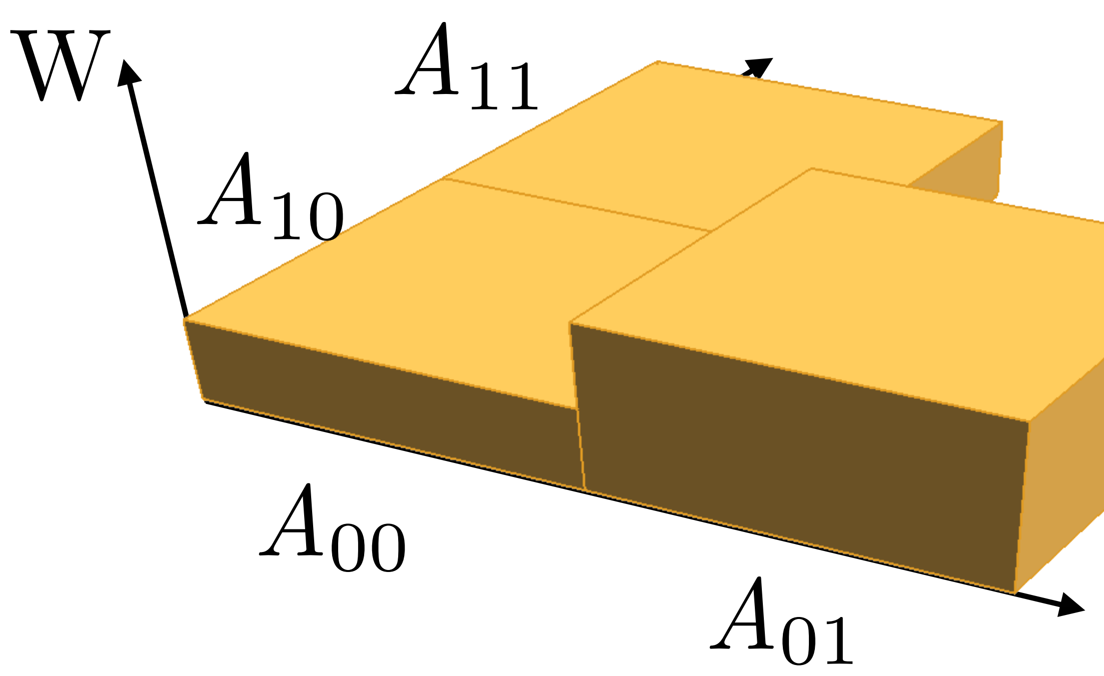
$$M_2 = -\imath(|0\rangle\langle 1| - |1\rangle\langle 0|)$$

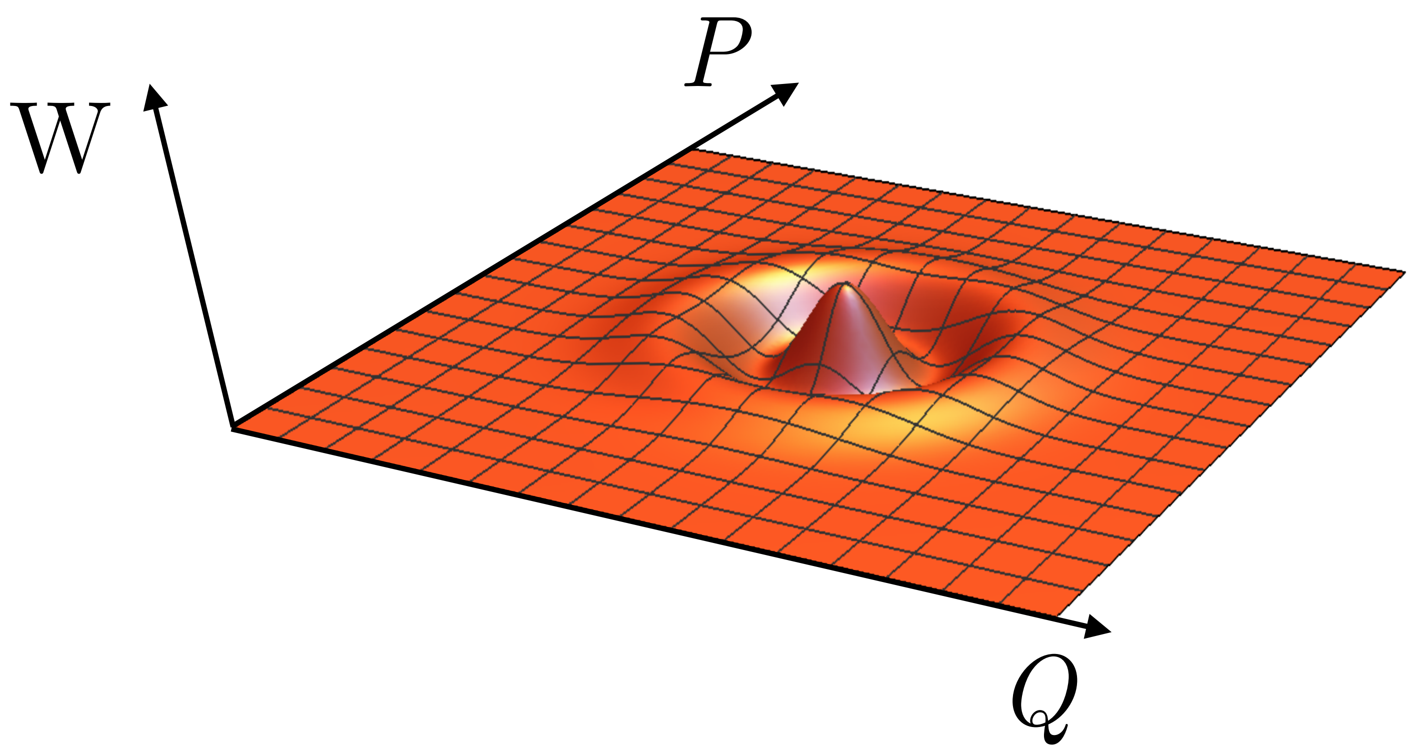
$$M_3 = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Pauli Matrices

Reconstruction of a Quantum State

- It is **impossible to determine the unknown quantum state of a single individual system** by any measurement or sequence of measurements. We need to perform statistics on an ensemble of quantum systems, i.e., measuring different observables.





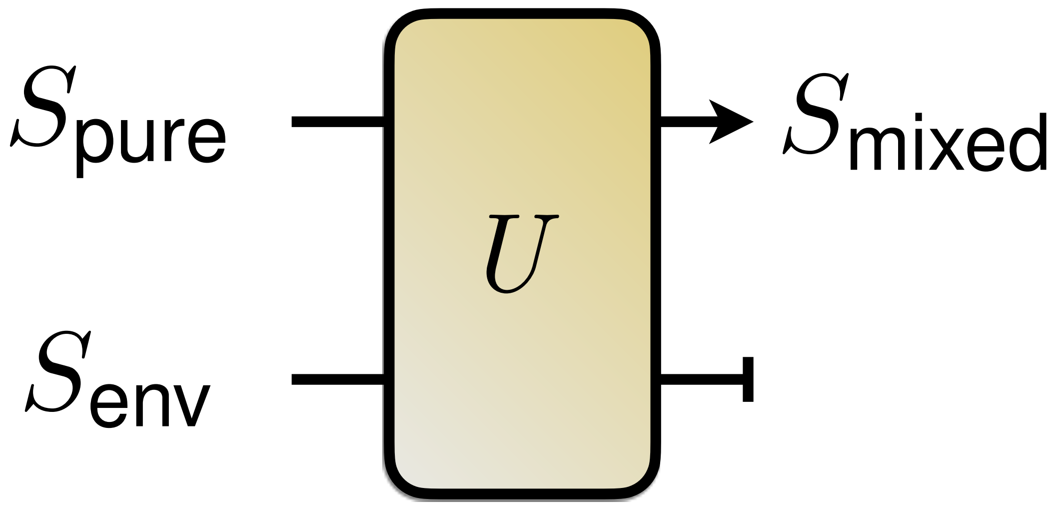
$$\mathrm{W}_{ij} = \frac{1}{2}\mathrm{tr}(S A_{ij})$$

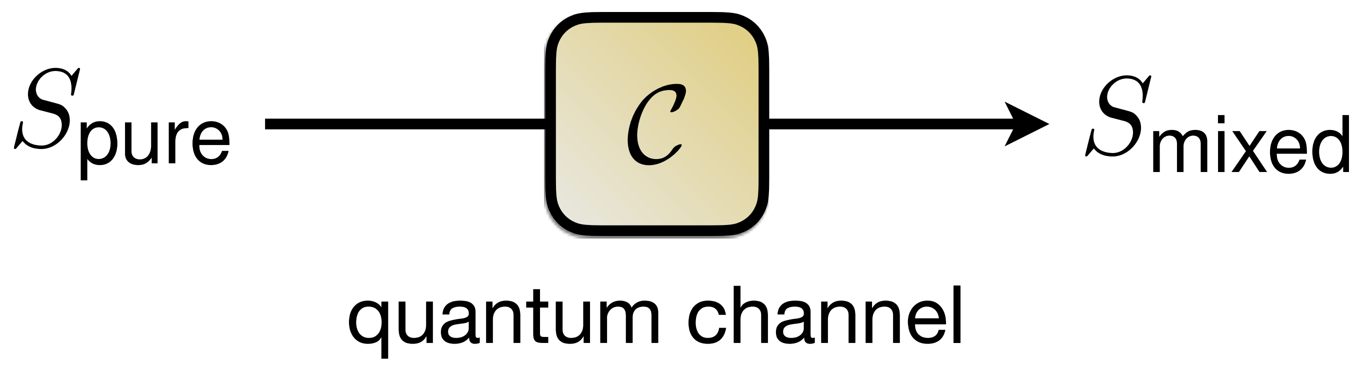
$$\mathrm{W} = \frac{1}{4\pi}\int_{-\infty}^{\infty} dx \exp\left\{\frac{\imath x p}{2}\right\} \left\langle q - \frac{x}{2} \middle| S \middle| q + \frac{x}{2} \right\rangle$$

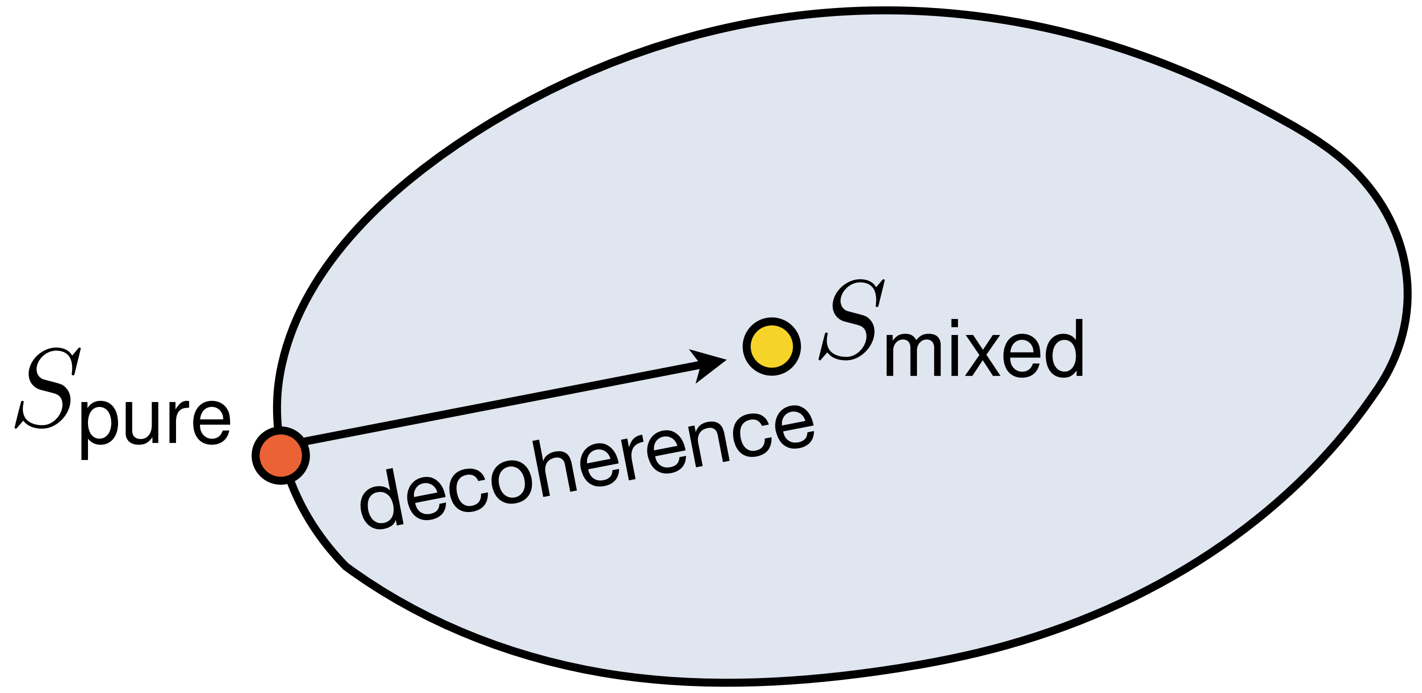
- The process of reconstructing the quantum state is called **quantum tomography**

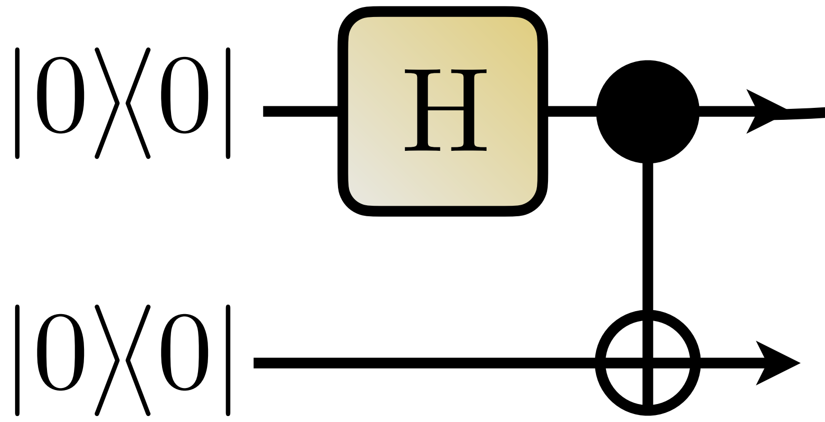
D. T. Smithey et al., Phys. Rev. Lett. (1993)
M. G. Raymer et al., Phys. Rev. Lett. (1994)

Noise Leads to Decoherence









$$\Phi = \frac{|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|}{2}$$

$$\mathrm{tr}_{\text{env}}(\Phi) = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}$$

- **Pure** quantum states satisfy $\mathrm{tr}\left(S^2\right) = 1$
- **Mixed** quantum states satisfy $\frac{1}{2} \leqslant \mathrm{tr}\left(S^2\right) < 1$

Purity is measured by $\mathrm{tr}\left(S^2\right)$

Noise decreases the **purity** of the quantum states

Entanglement (Revisited)

Uncorrelated

$$S^{(AB)} = S^{(A)} \otimes S^{(B)}$$

Correlated

$$S^{(AB)} \neq S^{(A)} \otimes S^{(B)}$$

Classically Correlated

$$S^{(AB)} = \sum_i p_i S_i^{(A)} \otimes S_i^{(B)}$$

Quantum Correlated

$$S^{(AB)} \neq \sum_i p_i S_i^{(A)} \otimes S_i^{(B)}$$

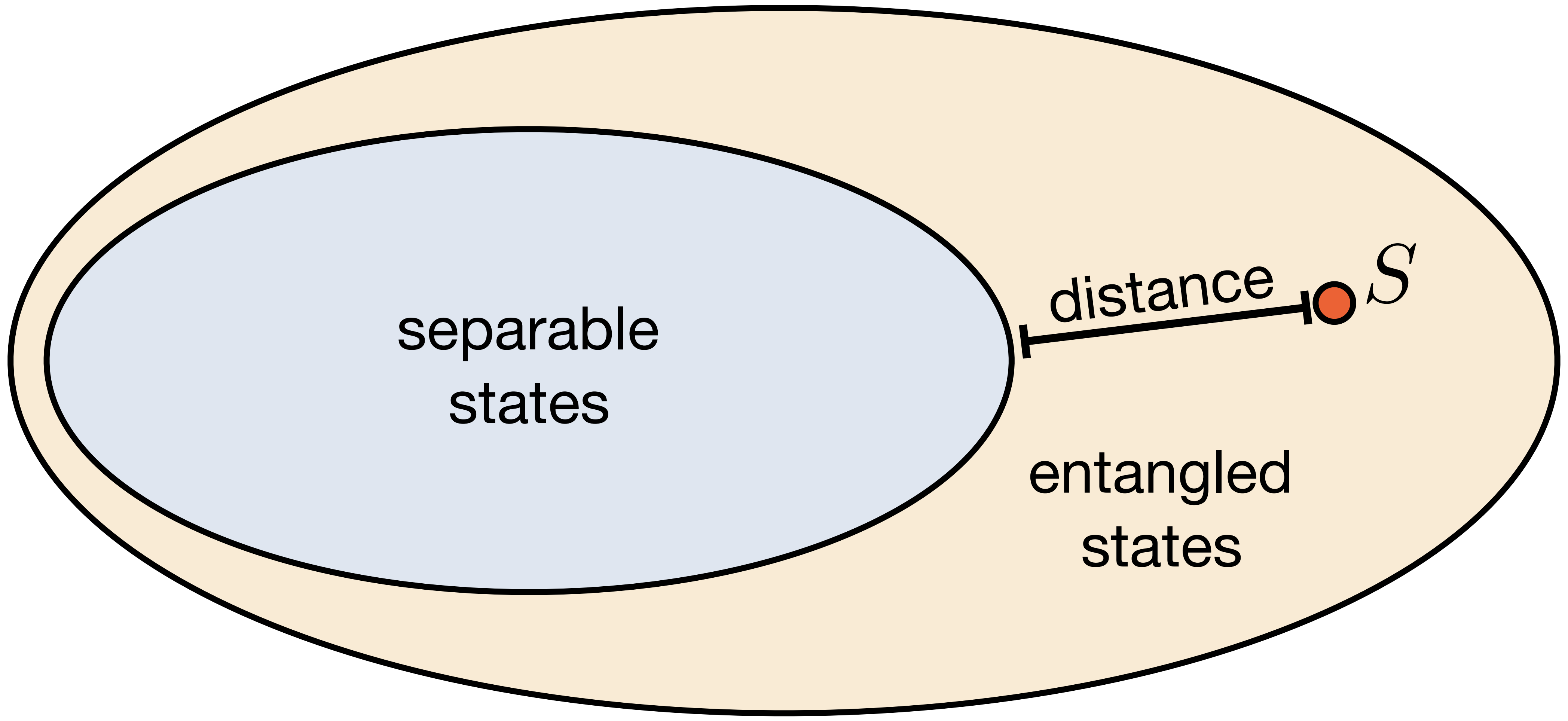
Separable

Entangled

- **Identifying** entanglement is hard
- **Quantifying** entanglement is even harder

Quantifying Entanglement

Geometrical Measures



- What is the **minimum distance** between an entangled state and the set of separable states?

Relative entropy of entanglement

$$\mathrm{E}_R(S) = \min_{S_{\text{sep}}} \mathbb{H}(S||S_{\text{sep}})$$

Operational Measures

- What is the **minimum amount of Bell states** used to **create** a state?

$$|\Phi\rangle^{\otimes n} \xrightarrow{\text{LOCC}} S^{\otimes m}$$

Entanglement cost

$$\mathrm{E}_C := \min_{\text{LOCC}} \left\{ \frac{n}{m} \right\}$$

- What is the **maximum amount of Bell states** we can **create** from a state?

$$S^{\otimes m} \xrightarrow{\text{LOCC}} |\Phi\rangle^{\otimes n}$$

Distillable entanglement

$$\mathrm{E}_D := \max_{\text{LOCC}} \left\{ \frac{n}{m} \right\}$$

Noise decreases the **entanglement** of the quantum states

R. Horodecki et al., Rev. Mod. Phys. (2009)

Lecture 3 — Decoherence and Quantum Networks

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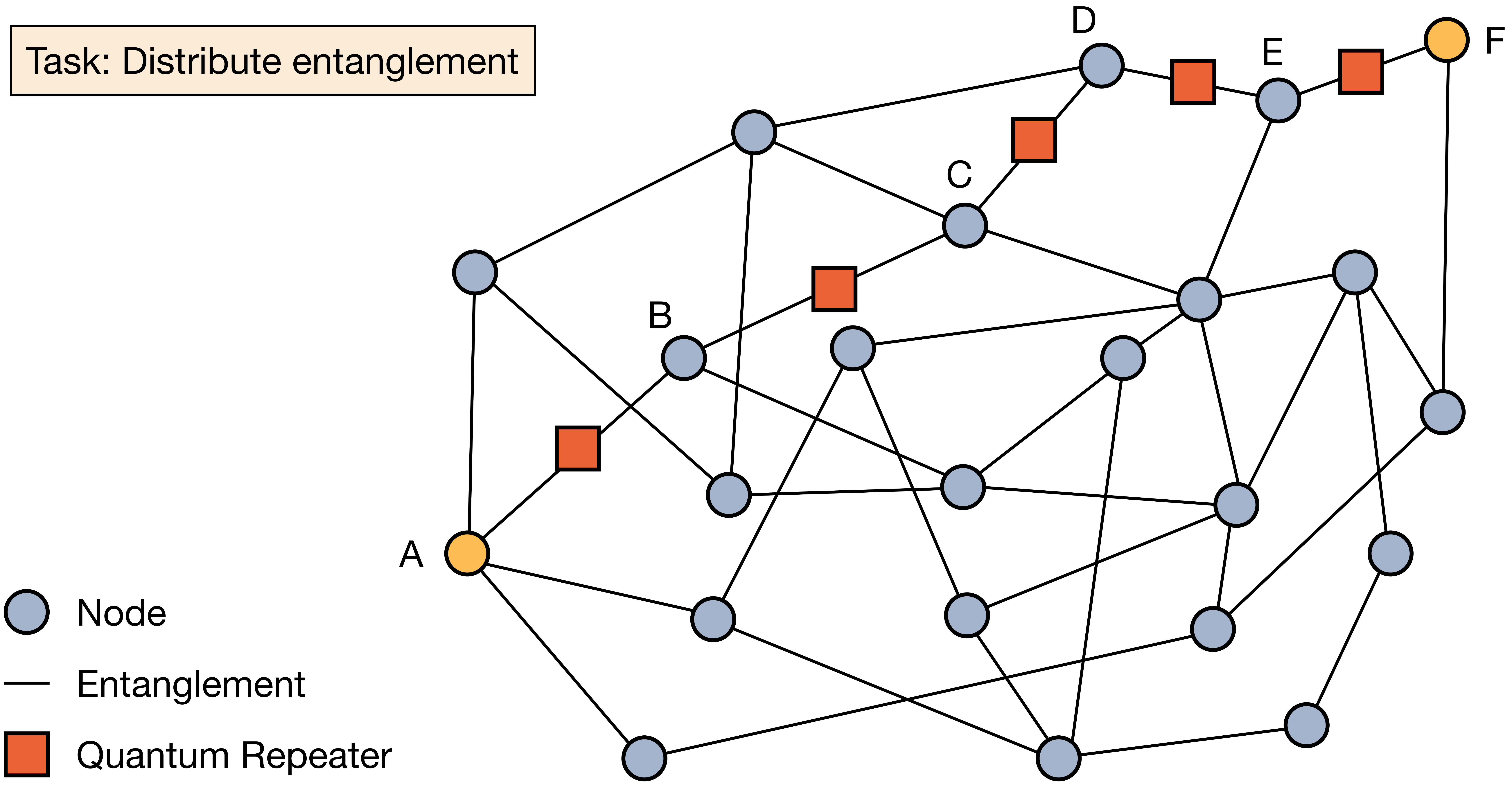
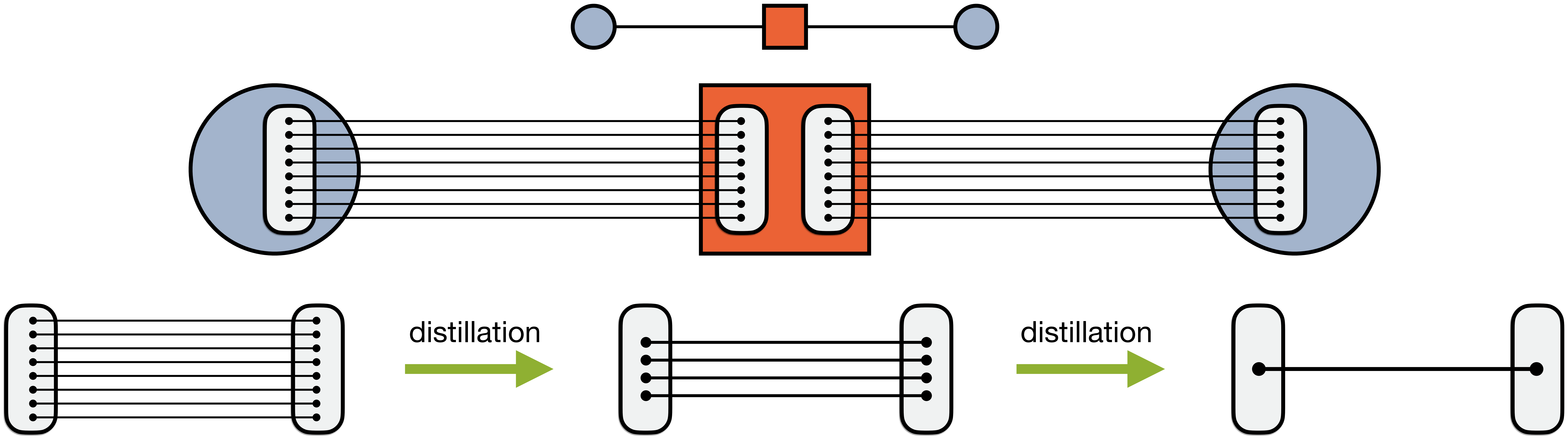
  - Quantification of Entanglement
- **Quantum Networks With Repeaters**
  - **Entanglement Distillation**
  - **Quantum Key Distribution**

Quantum Network with Repeaters

Task: Distribute entanglement



Quantum Repeater

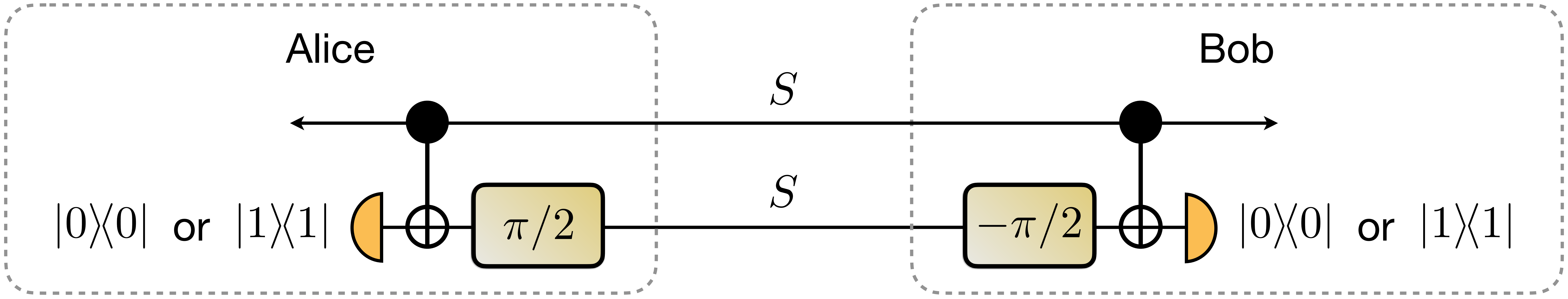


- **Entanglement distillation** (entanglement purification) is a probabilistic technique that creates a strongly entangled state out of a set of weakly entangled ones

- Quantum repeaters require **quantum memories** to store entanglement for future usage
- Quantum repeaters can also utilize **error correction codes** to further boost their performance

H.-J. Briegel, Phys. Rev. Lett. (1998)

Entanglement Distillation



Bell diagonal state

$$S = a\left|\Phi^+\right\rangle\left\langle\Phi^+\right| + b\left|\Psi^-\right\rangle\left\langle\Psi^-\right| + c\left|\Psi^+\right\rangle\left\langle\Psi^+\right| + d\left|\Phi^-\right\rangle\left\langle\Phi^-\right|$$

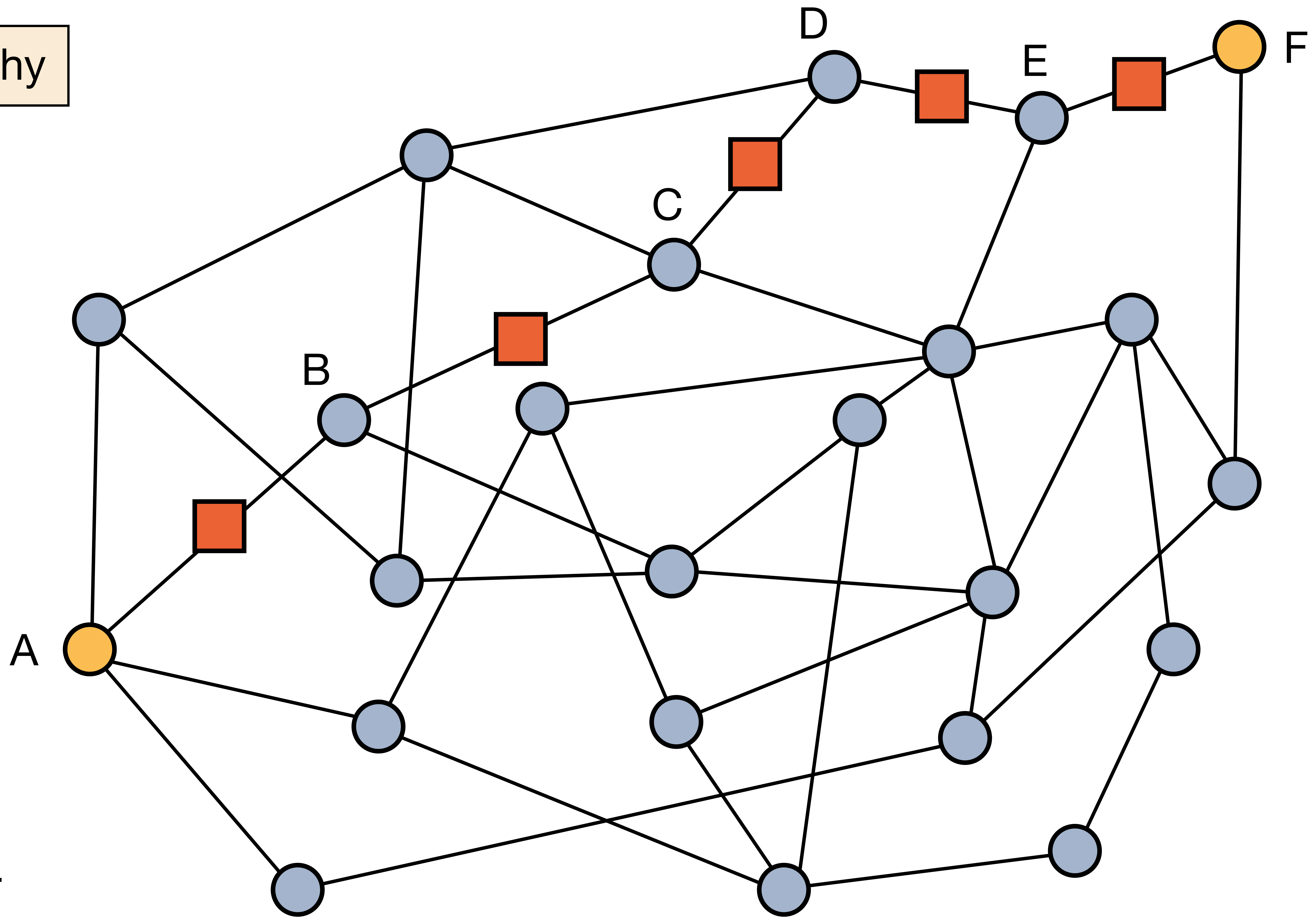
$$\left|\Phi^\pm\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$

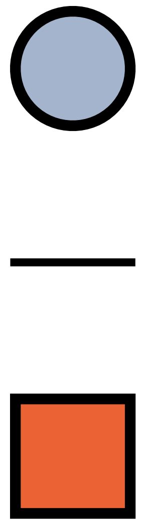
$$\left|\Psi^\pm\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$$

C. H. Bennett et al., Phys. Rev. Lett. (1996)
D. Deutsch et al., Phys. Rev. Lett. (1996)

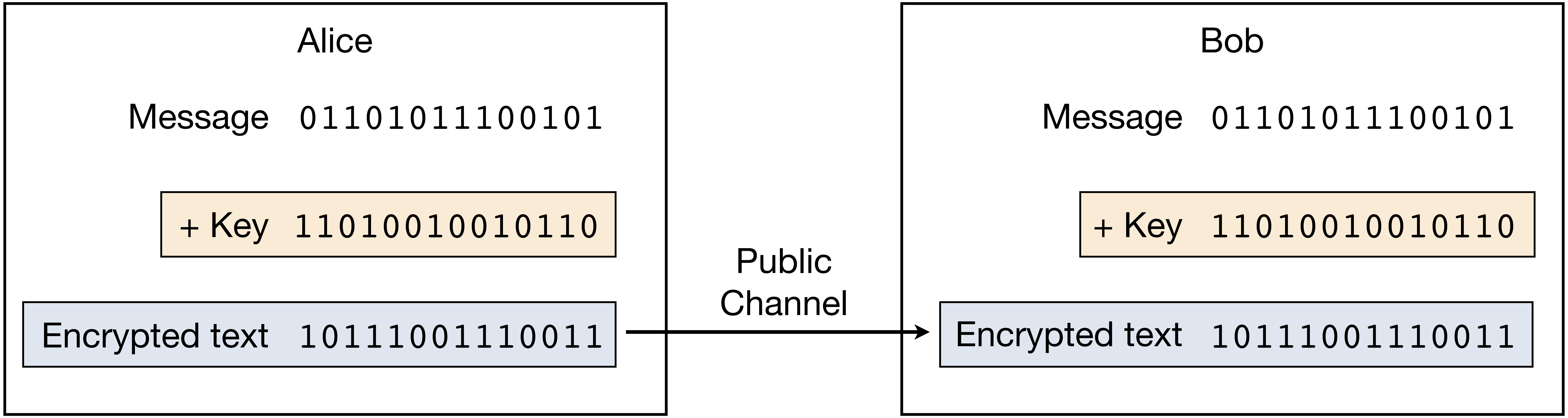
Applications of Quantum Networks

- Quantum cryptography
- Quantum computing
- Quantum sensing





Cryptography

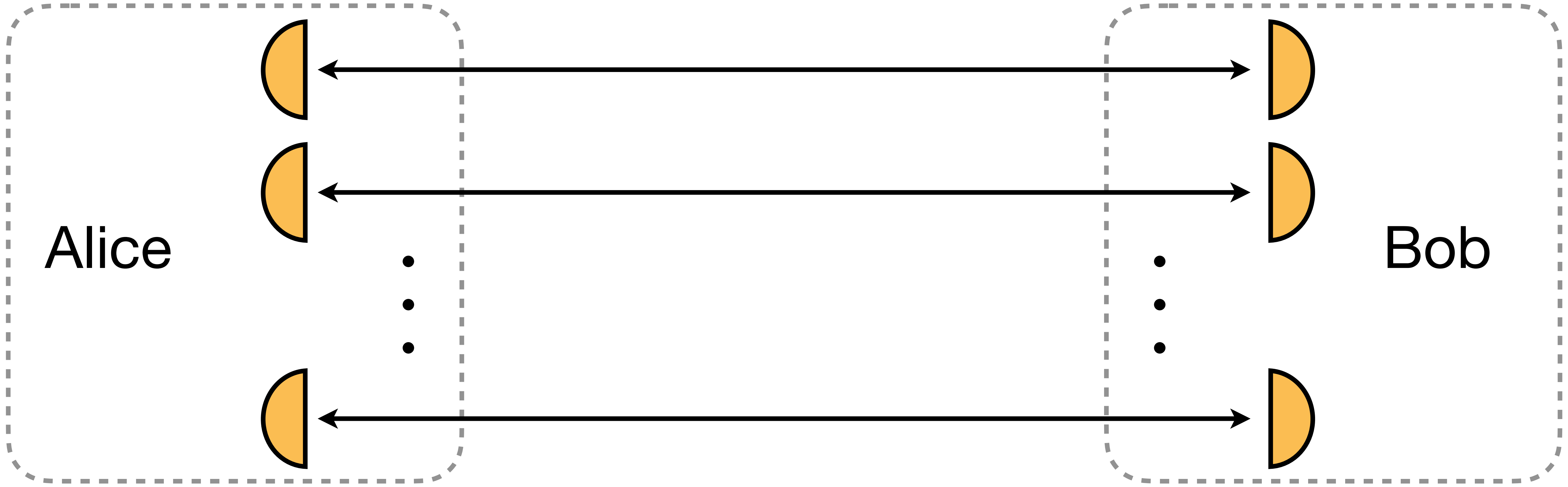


Secure communication if the key is:

- the same size as the message ✓
- used only once ✓
- random ✓
- securely distributed ? **Quantum Key Distribution**

C. E. Shannon, The Bell System Technical Journal (1949)

Quantum Key Distribution

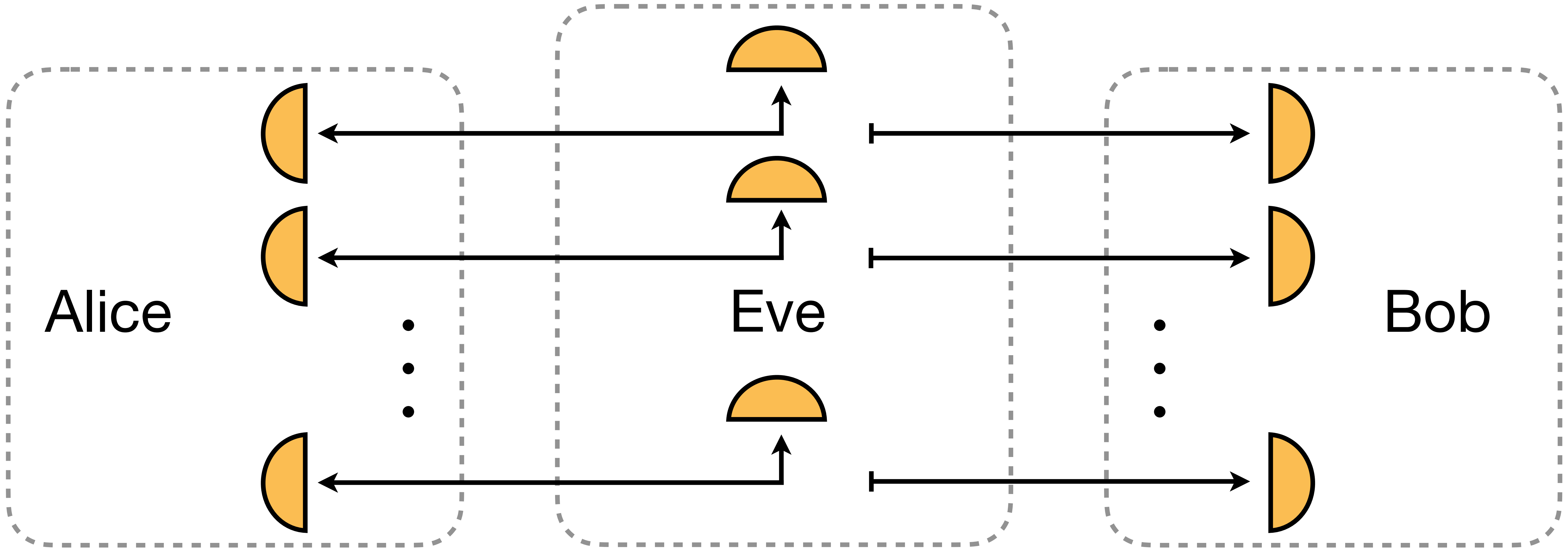


$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|\updownarrow\updownarrow\rangle + |\leftrightarrow\leftrightarrow\rangle) = \frac{1}{\sqrt{2}}(|\swarrow\!\!\!\nearrow\swarrow\!\!\!\nearrow\rangle + |\nwarrow\!\!\!\searrow\nwarrow\!\!\!\searrow\rangle)$$

State number	1	2	3	4	5	6	7	8	9	10
Alice's basis	✛	✛	✕	✛	✕	✛	✛	✕	✕	✛
Alice's observation	↔	↔	⤡	↕	⤢	↔	↔	⤡	⤢	↕
Bob's basis	✕	✛	✕	✛	✛	✕	✛	✛	✕	✕
Bob's observation	⤡	↔	⤡	↕	↕	⤢	↔	↕	⤢	⤡
		1	1	0			1		0	

A. K. Ekert, Phys. Rev. Lett. (1991)
C. H. Bennett, G. Brassard, N. D. Mermin, Phys. Rev. Lett. (1992)

Quantum Key Distribution



No Cloning Theorem

$$U\left(|\psi\rangle \otimes |\varphi\rangle\right) \neq |\psi\rangle \otimes |\psi\rangle$$

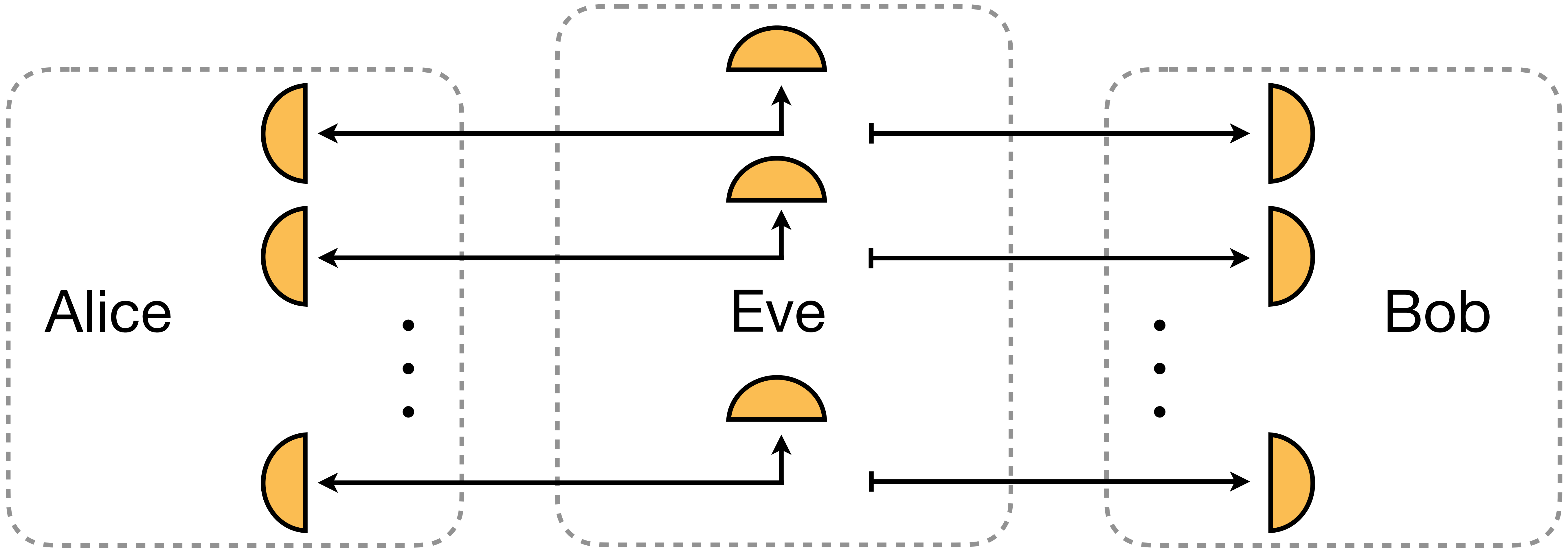
ancillary state

input state

J. L. Park, Found. Phys. (1970)

State number	1	2	3	4	5	6	7	8	9	10
Alice’s basis	+	+	×	+	×	+	+	×	×	+
Alice’s observation	↔	↔	↘	↕	↗	↔	↔	↘	↗	↕
Bob’s basis	×	+	×	+	+	×	+	+	×	×
Bob’s observation	↘	↔	↕	↘	↕	↗	↔	↔	↕	↕
Eve’s basis	×	+	+	×	+	×	+	+	+	+
Eve’s observation	↘	↔	↕	↘	↕	↗	↔	↔	↕	↕

Quantum Key Distribution



No Cloning Theorem

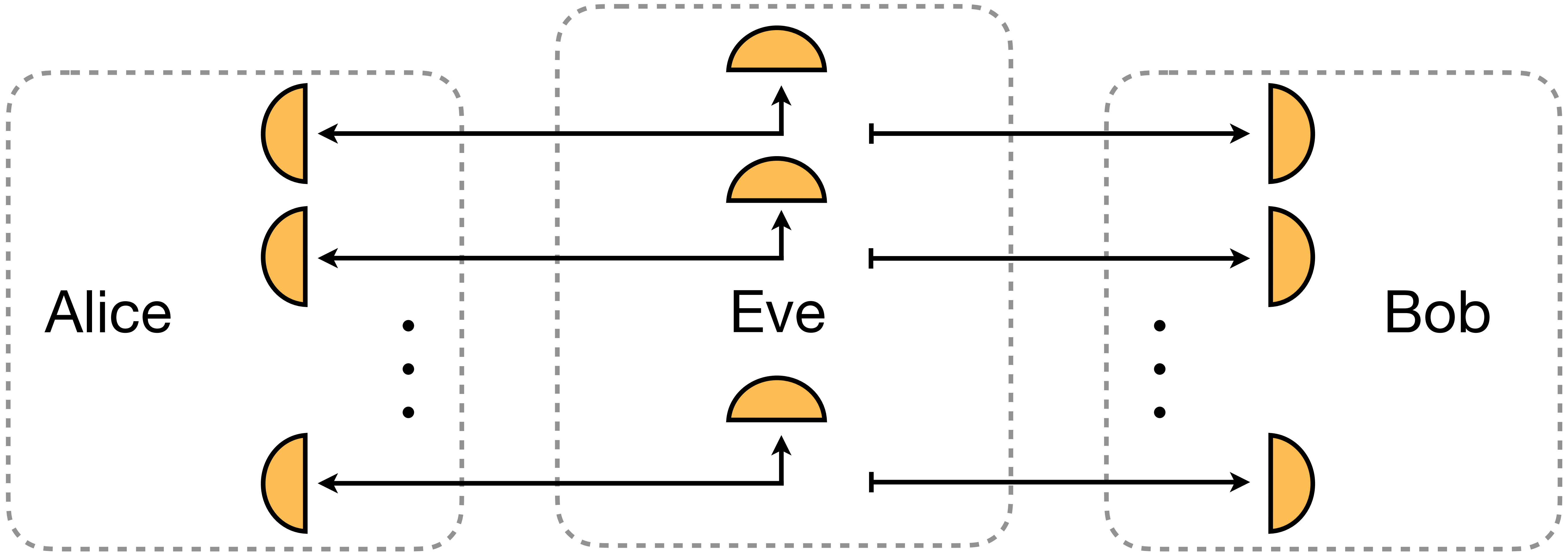
$$U\left(|\psi\rangle \otimes |\varphi\rangle\right) \neq |\psi\rangle \otimes |\psi\rangle$$

input state ($|\psi\rangle$)
ancillary state ($|\varphi\rangle$)

J. L. Park, Found. Phys. (1970)

State number	1	2	3	4	5	6	7	8	9	10
Alice's basis	+	+	×	+	×	+	+	×	×	+
Alice's observation	↔	↔	⤡	↕	⤢	↔	↔	⤡	⤢	↕
Bob's basis	×	+	×	+	+	×	+	+	×	×
Bob's observation	⤡	↔	⤢	↕	↕	⤢	↔	↔	⤡	⤡
Eve's basis	×	+	+	×	+	×	+	+	+	+
Eve's observation	⤡	↔	↕	⤡	↕	⤢	↔	↔	↕	↕

Quantum Key Distribution



No Cloning Theorem

$$U\left(|\psi\rangle \otimes |\varphi\rangle\right) \neq |\psi\rangle \otimes |\psi\rangle$$

input state

ancillary state

J. L. Park, Found. Phys. (1970)

State number	1	2	3	4	5	6	7	8	9	10
Alice's basis	+	+	×	+	×	+	+	×	×	+
Alice's observation	↔	↔	⤡	↕	⤢	↔	↔	⤡	⤢	↕
Bob's basis	×	+	×	+	+	×	+	+	×	×
Bob's observation	⤡	↔	⤢	↕	↕	⤢	↔	↔	⤡	⤡
Eve's basis	×	+	+	×	+	×	+	+	+	+
Eve's observation	⤡	↔	↕	⤡	↕	⤢	↔	↔	↕	↕

So, what is quantum mechanics? *Even though it was discovered by physicists, it's not a physical theory in the same sense as electromagnetism or general relativity. In the usual 'hierarchy of sciences' – with biology at the top, then chemistry, then physics, then math – quantum mechanics sits at a level between math and physics that I don't know a good name for.*

Basically, ***quantum mechanics is the operating system that other physical theories run on as application software*** *(with the exception of general relativity, which hasn't yet been successfully ported to this particular OS). There's even a word for taking a physical theory and porting it to this OS: 'to quantize'.*

—**Scott Aaronson** (Prof. of Computer Science at UT, Austin)

Suggested Bibliography

- Quantum Mechanics
    - J. Townsend - “A Modern Approach to Quantum Mechanics”
    - L. Ballentine - “Quantum Mechanics”
- Quantum Information
    - J. Audretsch - “Entangled Systems”
    - M. Nielsen, I. Chuang - “Quantum Computation and Quantum Information”

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